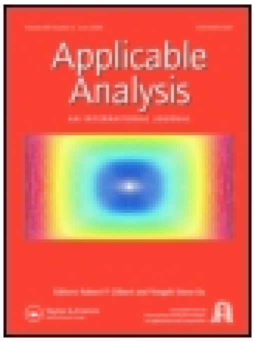

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ABSTRACT

In this paper, we address the analytical investigation into a model for adhesive contact introduced in a paper by Freddi and Fremond, which includes nonlocal sources of damage on the contact surface, such as the elongation. The resulting PDE system features various nonlinearities rendering the unilateral contact conditions, the physical constraints on the internal variables, as well as the contributions related to the nonlocal forces. For the associated initial-boundary value problem, we obtain a *global-in-time* existence result by proving the existence of a local solution via a suitable approximation procedure and then by extending the local solution to a global one by a nonstandard prolongation argument.

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1. Introduction

The mathematical field of contact mechanics has flourished over the last decades, as illustrated for instance in the monographs [1–3]. In this paper, we focus on a PDE system pertaining to a subclass of contact models, i.e. models for *adhesive contact*. This phenomenon plays an important role in the analysis of the stability of structures. Indeed, it is well known that interfacial regions between materials are fundamental to ensure the strength and stability of structural elements. We describe adhesive contact using a 'surface damage theory', as proposed by M. FRÉMOND (see e.g. [4]), for the action of the adhesive substance (one may think of glue), located on the contact surface. This 'damage approach' provides an efficient and predictive theory for the mechanical behavior of the structure.

The main idea underlying this modeling perspective is that while the basic unilateral contact theory does not allow for any resistance to tension, in adhesive contact resistance to tension is related to the action of microscopic bonds between the surfaces of the adhering solids. The resulting description of the state of the bonds between the two bodies is in the spirit of damage models. In fact, it is given in terms of a phase parameter χ akin to a damage parameter, characterizing the state of the cohesive bonds. The PDE system is recovered from the balance laws of continuum mechanics, including (in a generalized principle of virtual power) the effects of micro-forces and micro-motions which are responsible for the breaking (or possibly repairing) of the microscopic bonds on the contact surface.

The mathematical analysis of models for adhesive contact and delamination à la Frémond, pioneered by [5], has attracted remarkable attention over the last 15 years, both in the case of *rate-independent* (cf., e.g. [6–8]) and *rate-dependent* (see, among others, [9,10]) evolution.

The type of model investigated in the present paper was first rigorously derived and analyzed in [11] (cf. also [12]). The associated PDE system couples an equation for the macroscopic deformations of the body and a 'boundary' equation on the contact surface, describing the evolution of the state of the glue in terms of a surface damage parameter. The system is highly nonlinear, mainly due to the

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presence of nonlinear boundary conditions and nonsmooth constraints on the internal variables, providing the unilaterality of the contact, the physical consistency of the damage parameter and, possibly, the unidirectional character of the degradation process. The model from [11] has been generalized to the non-isothermal case, with the temperature evolution described by an entropy balance equation (cf. e.g. [13]) or by the more classical energy balance equation (see [14]). Furthermore, in [14–16] friction effects have also been included, even in the temperature-dependent case. Therefore, we deal with the coupling between the Signorini condition for adhesive contact and a nonlocal regularization of the Coulomb law where the friction coefficient may depend on the temperature.

In this paper, we again focus on the isothermal and frictionless case. In fact, we address the analysis of a model for adhesive contact that generalizes that from [11] by assuming that also nonlocal forces act on the contact surface. This leads to the presence of novel nonlocal (nonlinear) contributions in the resulting PDE system. More precisely, on the contact surface we consider interactions between damage at a point and damage in its neighborhood (i.e. we use a gradient surface damage theory) and moreover we admit an interaction between the adhesive substance and the two bodies. An example of this kind of behavior is given by experiments showing that elongation, i.e. a variation of the distance of two distinct points on the contact surface, may have damaging effects. Thus, a nonlocal quantity corresponding to the elongation is considered in the energy functional.

This model was introduced in [4,17]. While referring to the latter paper for details of the modeling and comments on the applications and computational results, in the following lines we will outline the derivation of the model for the sake of completeness. We will confine the discussion to the reduced case when only one body is considered in adhesive contact with a rigid support on a part of its boundary. We observe that this choice has the advantage of simplifying the exposition in comparison to the two-body case, while affecting neither the relevance of the model nor its analytical investigation.

1.1. The model and the PDE system

Let us consider a body which is located in a sufficiently smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and lying on a rigid support during a finite time interval $(0, T)$. We denote its boundary by $\partial\Omega = \overline{\Gamma}_{\mathrm{D}} \cup \overline{\Gamma}_{\mathrm{N}} \cup \overline{\Gamma}_{\mathrm{C}}$. Here Γ_{D}, Γ_{N}, Γ_{C} are open subsets in the relative topology of $\partial\Omega$, each of them with a smooth boundary and disjoint one from each other. In particular, Γ_{C} is the contact surface, which is considered as a flat surface and identified with a subset of $\mathbb{R}^2$, cf. (2.1) ahead. Hereafter we shall suppose that both Γ_{C} and Γ_{D} have positive measure. We let $\mathbf{u} = \mathbf{0}$ on Γ_{D}, while a known traction is applied on Γ_{N}.

As already mentioned, in the model we neither encompass thermal evolution nor frictional effects. Thus, the state variables of the model are

$$(\epsilon(\mathbf{u}), \chi, \nabla\chi, \mathbf{u}_{|\Gamma_{\mathrm{C}}}, \mathbf{g})$$

where we denote $\mathbf{u}$ the vector of small displacements, $\epsilon(\mathbf{u})$ the symmetric linearized strain tensor, χ a damage parameter defined on the contact surface, $\nabla\chi$ its gradient, and $\mathbf{u}_{|\Gamma_{\mathrm{C}}}$ the trace of the displacement $\mathbf{u}$ on Γ_{C}. The parameter χ is assumed to take values in $[0, 1]$, with $\chi = 0$ for completely damaged bonds, $\chi = 1$ for undamaged bonds, and $\chi \in (0, 1)$ for partially damaged bonds.

The nonlocal term $\mathbf{g}$, defined for $(x, y) \in \Gamma_{\mathrm{C}} \times \Gamma_{\mathrm{C}}$ by

$$\mathbf{g}(x, y) = 2(x - y)\mathbf{u}_{|\Gamma_{\mathrm{C}}}(x), \tag{1.1}$$

describes the damaging effects due to the elongation. The free energy of the system is given by the sum of a bulk, a local surface, and a nonlocal surface contributions. More precisely, the free energy (density) in Ω is the classical one in elasticity theory

$$\Psi_\Omega(\epsilon(\mathbf{u})) = \frac{1}{2}\epsilon(\mathbf{u})\mathbb{K}\epsilon(\mathbf{u}), \tag{1.2}$$

where $\mathbb{K} = (a_{ijkh})$ stands for the elasticity tensor. Moreover, the local surface part of the free energy (density) is given by

$$\Psi_{\Gamma_C}(\chi, \nabla\chi, \mathbf{u}_{|\Gamma_C}) = I_{[0,1]}(\chi) + \gamma(\chi) + \frac{1}{2}|\nabla\chi|^2 + \frac{1}{2}\chi|\mathbf{u}_{|\Gamma_C}|^2 + I_{(-\infty,0]}(\mathbf{u}_{|\Gamma_C}\cdot\mathbf{n}), \tag{1.3}$$

where the indicator function $I_{[0,1]}$ of the interval $[0, 1]$ accounts for a physical constraint on χ, being $I_{[0,1]}(\chi) = 0$ if $\chi \in [0, 1]$ and $I_{[0,1]}(\chi) = +\infty$ otherwise. Analogously, denoting by $I_{(-\infty,0]}$ the indicator function of the interval $(-\infty, 0]$, the term $I_{(-\infty,0]}(\mathbf{u}_{|\Gamma_C}\cdot\mathbf{n})$ renders the impenetrability condition on the contact surface, as it enforces that $\mathbf{u}_{|\Gamma_C}\cdot\mathbf{n} \leq 0$ ($\mathbf{n}$ is the outward unit normal vector to Γ_C). Finally, the function γ, sufficiently smooth and possibly nonconvex, is related to non-monotone dynamics for χ (from a physical point of view, it includes some cohesion in the material). Finally, let us consider the nonlocal surface free energy (density), which is given by

$$\Psi_{nl}(\chi(x), \chi(y), \mathbf{g}(x, y)) = \frac{1}{2}\mathbf{g}^2(x, y)\chi(x)\chi(y)e^{\frac{-|x-y|^2}{d^2}}, \tag{1.4}$$

where the exponential function with distance d describes the attenuation of nonlocal actions with distance $|x - y|$ between points x and y on the contact surface.

As far as dissipation, we assume that there is no dissipation due to changes of the nonlocal variable while we consider the dissipative variables $\epsilon(\mathbf{u}_t)$ in Ω and χ_t in Γ_C. We follow the approach proposed by J. J. MOREAU to prescribe the dissipated energy by means of the so-called pseudo-potential of dissipation which is a convex, nonnegative functional, attaining its minimal value 0 when the dissipation (described by the dissipative variables) is zero. More precisely, we define the volume part Φ_Ω of the pseudo-potential of dissipation by

$$\Phi_\Omega(\epsilon(\mathbf{u}_t)) = \frac{1}{2}\epsilon(\mathbf{u}_t)\mathbb{K}_\nu\epsilon(\mathbf{u}_t), \tag{1.5}$$

where $\mathbb{K}_\nu = (b_{ijkh})$ denotes the viscosity tensor. The surface part Φ_{Γ_C} of the pseudo-potential of dissipation depends on χ_t via

$$\Phi_{\Gamma_C}(\chi_t) = \frac{1}{2}|\chi_t|^2 + I_{(-\infty,0]}(\chi_t). \tag{1.6}$$

The quadratic term $\frac{1}{2}|\chi_t|^2$ encodes *rate-dependent* evolution of χ, where the indicator term $I_{(-\infty,0]}(\chi_t)$ forces χ_t to take nonpositive values and renders the unidirectional character of the damage process.

Hereafter, we shall omit for simplicity the index $v_{|\Gamma_C}$ to denote the trace on Γ_C of a function v, defined in Ω. The equations are recovered by a generalization of the principle of virtual powers, in which microscopic forces (also nonlocal ones) responsible for the damage process in the adhesive substance are included. More precisely, for any virtual bulk velocity $\mathbf{v}$ with $\mathbf{v} = \mathbf{0}$ on Γ_D and for any virtual microscopic velocity w on the contact surface, we define the power of internal forces in Ω and Γ_C as follows

$$\begin{aligned}\mathcal{P}_{int} = &-\int_\Omega \Sigma\epsilon(\mathbf{v})\,\mathrm{d}\Omega - \int_{\Gamma_C}(Bw + \mathbf{H}\nabla w)\,\mathrm{d}x + \int_{\Gamma_C}\mathbf{R}\mathbf{v}\,\mathrm{d}x\\ &+ \int_{\Gamma_C}\int_{\Gamma_C} 2M(x,y)(x-y)\mathbf{v}(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_C}\int_{\Gamma_C}(B^1_{nl}(x,y)w(x) + B^2_{nl}(x,y)w(y))\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{1.7}$$

Here, Σ is the Cauchy stress tensor, $\mathbf{R}$ is the classical macroscopic reaction on the contact surface, B and $\mathbf{H}$ are local interior forces, responsible for the degradation of the adhesive bonds between the body and the support. The terms $M(x,y)$ and $B^i_{\rm nl}(x,y)$, $i=1,2$, are new scalar nonlocal contributions: they stand for internal microscopic nonlocal forces on the contact surface and describe the effects of the elongation as a source of damage. The power of the external forces is given by

$$\mathcal{P}_{ext} = \int_\Omega \mathbf{f}\mathbf{v}\,\mathrm{d}\Omega + \int_{\Gamma_{\rm N}} \mathbf{h}\mathbf{v}\,\mathrm{d}\Gamma, \tag{1.8}$$

where $\mathbf{f}$ is a bulk known external force, while $\mathbf{h}$ is a given traction on $\Gamma_{\rm N}$. Note that, here we have disregarded external forces acting on the microscopic level and confined ourselves to the case of null accelerations power.

The principle of virtual powers, holding for every virtual microscopic and macroscopic velocities and every subdomain in Ω, leads to the quasistatic momentum balance

$$-\operatorname{div}\Sigma = \mathbf{f} \quad \text{in } \Omega, \tag{1.9}$$

which will be also posed in a time interval $(0,T)$, and supplemented by the following boundary conditions

$$\Sigma\mathbf{n}(x) = \mathbf{R}(x) + \int_{\Gamma_{\rm C}} 2(x-y)M(x,y)\,\mathrm{d}y \quad \text{in } \Gamma_{\rm C}, \tag{1.10}$$

$$\mathbf{u}=\mathbf{0} \quad \text{in } \Gamma_{\rm D}, \qquad \Sigma\mathbf{n}=\mathbf{h} \quad \text{in } \Gamma_{\rm N}. \tag{1.11}$$

Observe that in (1.10) the boundary condition for the stress tensor on the contact surface combines a local contribution involving the (pointwise) reaction $\mathbf{R}(x)$ and a nonlocal force (defined in terms of the new variable $M(x,y)$), related to the elongation.

Again, the principle of virtual powers leads to a micro-force balance on the contact surface, also posed in the time interval $(0,T)$, given by

$$B(x) - \operatorname{div}\mathbf{H}(x) = \int_{\Gamma_{\rm C}} \left(B^1_{\rm nl}(x,y) + B^2_{\rm nl}(y,x)\right)\,\mathrm{d}y \ \text{ in } \Gamma_{\rm C}, \qquad \mathbf{H}\cdot\mathbf{n}_s = 0 \ \text{ on } \partial\Gamma_{\rm C}. \tag{1.12}$$

Here, $\mathbf{n}_s$ denotes the outward unit normal vector to $\partial\Gamma_{\rm C}$.

Constitutive relations for $\Sigma, \mathbf{R}, B, \mathbf{H}, M$, and $B^i_{\rm nl}$, $i=1,2$, are given in terms of the free energies and the pseudo-potentials of dissipation. More precisely, the constitutive relation for the stress tensor Σ accounts for the dissipative (viscous) dynamics of the deformations

$$\Sigma = \frac{\partial\Psi_\Omega}{\partial\epsilon(\mathbf{u})} + \frac{\partial\Phi_\Omega}{\partial\epsilon(\mathbf{u}_t)} = \mathbb{K}\epsilon(\mathbf{u}) + \mathbb{K}_v\epsilon(\mathbf{u}_t), \tag{1.13}$$

while the local reaction $\mathbf{R}$ is given by

$$\mathbf{R} = -\frac{\partial\Psi_{\Gamma_{\rm C}}}{\partial\mathbf{u}} \in -\chi\mathbf{u} - \partial I_{(-\infty,0]}(\mathbf{u}\cdot\mathbf{n})\mathbf{n}. \tag{1.14}$$

As for as the nonlocal force on $\Gamma_{\rm C}$, we prescribe

$$M(x,y) = -\frac{\partial\Psi_{\rm nl}}{\partial\mathbf{g}(x,y)} = -\mathbf{g}(x,y)\chi(x)\chi(y)\mathrm{e}^{-\frac{|x-y|^2}{d^2}}, \tag{1.15}$$

while the terms $B^1_{\rm nl}$ and $B^2_{\rm nl}$ are (formally) defined as derivatives of $\Psi_{\rm nl}$ with respect to the values of the surface damage parameter in x and $y \in \Gamma_{\rm C}$, respectively, as it follows

$$B^1_{\rm nl}(x,y) = -\frac{\partial \Psi_{\rm nl}}{\partial \chi(x)} = -\frac{\mathbf{g}^2(x,y)}{2} \mathrm{e}^{-\frac{|x-y|^2}{d^2}} \chi(y), \tag{1.16}$$

$$B^2_{\rm nl}(x,y) = -\frac{\partial \Psi_{\rm nl}}{\partial \chi(y)} = -\frac{\mathbf{g}^2(x,y)}{2} \mathrm{e}^{-\frac{|x-y|^2}{d^2}} \chi(x). \tag{1.17}$$

We further prescribe B by

$$B = \frac{\partial \Psi_{\Gamma_{\rm C}}}{\partial \chi} + \frac{\partial \Phi_{\Gamma_{\rm C}}}{\partial \chi_t} \in \partial I_{[0,1]}(\chi) + \gamma'(\chi) + \frac{1}{2}|\mathbf{u}|^2 + \chi_t + \partial I_{(-\infty,0]}(\chi_t), \tag{1.18}$$

($\partial I_{(-\infty,0]}$ and $\partial I_{[0,1]}$ denoting the convex analysis subdifferentials of $I_{(-\infty,0]}$ and $I_{[0,1]}$, resp.), and let $\mathbf{H}$ be

$$\mathbf{H} = \frac{\partial \Psi_{\Gamma_{\rm C}}}{\partial \nabla \chi} = \nabla \chi. \tag{1.19}$$

Combining the previous constitutive relations with the balance laws, we obtain the following boundary value problem

$$-\operatorname{div}\left(\mathbb{K}\epsilon(\mathbf{u}(x,t)) + \mathbb{K}_v\epsilon(\mathbf{u}_t(x,t))\right) = \mathbf{f}(x,t), \qquad (x,t) \in \Omega \times (0,T), \tag{1.20a}$$

$$\mathbf{u}(x,t) = \mathbf{0}, \qquad (x,t) \in \Gamma_{\rm D} \times (0,T), \tag{1.20b}$$

$$(\mathbb{K}\epsilon(\mathbf{u}(x,t)) + \mathbb{K}_v\epsilon(\mathbf{u}_t(x,t)))\mathbf{n} = \mathbf{h}(x,t), \qquad (x,t) \in \Gamma_{\rm N} \times (0,T), \tag{1.20c}$$

$$\begin{aligned}&(\mathbb{K}\epsilon(\mathbf{u}(x,t)) + \mathbb{K}_v\epsilon(\mathbf{u}_t(x,t)))\mathbf{n} + \chi(x,t)\mathbf{u}(x,t) + \partial I_{(-\infty,0]}(\mathbf{u}(x,t)\cdot\mathbf{n})\mathbf{n} \\ &\quad + \int_{\Gamma_{\rm C}} 2(x-y)\mathbf{g}(x,y)\chi(x,t)\chi(y,t)\mathrm{e}^{-\frac{|x-y|^2}{d^2}}\,\mathrm{d}y \ni \mathbf{0}, \qquad (x,t) \in \Gamma_{\rm C} \times (0,T),\end{aligned} \tag{1.20d}$$

$$\begin{aligned}&\chi_t(x,t) + \partial I_{(-\infty,0]}(\chi_t(x,t)) - \Delta\chi(x,t) + \partial I_{[0,1]}(\chi(x,t)) + \gamma'(\chi(x,t)) \\ &\ni -\frac{1}{2}|\mathbf{u}(x,t)|^2 - \frac{1}{2}\int_{\Gamma_{\rm C}} \left(\mathbf{g}^2(x,y) + \mathbf{g}^2(y,x)\right)\chi(y,t)\mathrm{e}^{-\frac{|x-y|^2}{d^2}}\,\mathrm{d}y, \qquad (x,t) \in \Gamma_{\rm C} \times (0,T),\end{aligned} \tag{1.20e}$$

$$\partial_{\mathbf{n}_s}\chi(x,t) = 0 \qquad (x,t) \in \partial\Gamma_{\rm C} \times (0,T), \tag{1.20f}$$

where all integrals on $\Gamma_{\rm C}$ involve the Lebesgue measure, which coincides with the Hausdorff measure on $\Gamma_{\rm C}$ by the flatness requirement, cf. (2.1) ahead.

Let us stress once again, with respect to the 'standard' Frémond's system for adhesive contact, (1.20) encompasses nonlocal terms both in the normal reaction on the contact surface (cf. with (1.20d)) and in the flow rule (1.20e) for χ. In particular, we note that the right-hand side of (1.20e) (see also the second of (1.21)) may be different from zero even if $\mathbf{u}(x,t) = \mathbf{0}$, due to the integral contributions which render the damaging effects of the elongation.

Taking into account the explicit expression (1.1) of $\mathbf{g}$, observe the integral terms on the left-hand side of (1.20d) and on the right-hand side of (1.20e) can be rewritten as

$$\begin{cases} \chi(x,t)\mathbf{u}(x,t)\int_{\Gamma_C}\eta(|x-y|)\chi(y,t)\,dy, \\ -\frac{1}{2}|\mathbf{u}|^2(x,t)\int_{\Gamma_C}\eta(|x-y|)\chi(y,t)\,dy-\frac{1}{2}\int_{\Gamma_C}\eta(|x-y|)\chi(y,t)|\mathbf{u}|^2(y,t)\,dy \end{cases} \text{with } \eta(\zeta)=4\zeta^2 e^{-\zeta^2/d^2}. \tag{1.21}$$

In view of (1.21), it is natural to address the analysis of a *generalization* of system (1.20), cf. (2.16) ahead, where the kernel $\eta(|x-y|)$, featuring the bounded, even, and positive function $\eta \in C^0(\mathbb{R})$, is replaced by a bounded, symmetric, and positive kernel $k : \Gamma_C \times \Gamma_C \to [0,\infty)$, inducing the *nonlocal* operator

$$\mathcal{K}[w](x) := \int_{\Gamma_C} k(x,y)w(y)\,dy, \quad w \in L^1(\Gamma_C). \tag{1.22}$$

1.2. Analytical results and plan of the paper

Our main result, **Theorem 2.1**, states the existence of global-in-time solutions for the Cauchy problem associated with a *generalized* version of system (1.20), featuring the nonlocal operator $\mathcal{K}$, and where the various, concrete, subdifferential operators are replaced by maximal monotone nonlinearities. It is stated in Section 2, where we collect all the assumptions on the problem data and introduce a suitable variational formulation of our PDE system. In Section 3, we set up a suitable approximation, for which we prove a local-in-time well-posedness result by means of Schauder fixed point technique. In Section 4, by a priori estimates combined with compactness and monotonicity tools, we develop a passage to the limit argument and we obtain a local-in-time solution for the original problem. Finally, in Section 5 we complete the proof of Theorem 2.1, extending the local solution to a global one by a carefully devised prolongation procedure.

2. Setup and main result

Throughout the paper, we shall assume that

$$\begin{gathered} \Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^3, \text{ with} \\ \partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C,\ \Gamma_D,\Gamma_N,\Gamma_C, \text{ open disjoint subsets in the relative topology of } \partial\Omega, \text{ such that} \\ \mathcal{H}^2(\Gamma_D),\ \mathcal{H}^2(\Gamma_C)>0, \qquad \text{and } \Gamma_C \subset \mathbb{R}^2 \text{ a sufficiently smooth } \textit{flat} \text{ surface.} \end{gathered} \tag{2.1}$$

More precisely, by *flat* we mean that Γ_C is a subset of a hyperplane of $\mathbb{R}^3$ and on Γ_C the Lebesgue and Hausdorff measures $\mathcal{L}^2$ and $\mathcal{H}^2$ coincide. As for smoothness, we require that Γ_C has a $C^{1,1}$-boundary.

Hereafter, we will use the following

Notation 2.1: Given a Banach space X, we denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between its dual space X' and X itself and by $\|\cdot\|_X$ both the norm in X and in any power of it. In particular, we shall use short-hand notation for some function spaces

$$\begin{gathered} \mathbf{H} := L^2(\Omega;\mathbb{R}^3), \quad \mathbf{V} := \{\mathbf{v} \in H^1(\Omega;\mathbb{R}^3) : \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_D\}, \quad \mathbf{Y} := H^{1/2}_{00,\Gamma_D}(\Gamma_C;\mathbb{R}^3), \\ H := L^2(\Gamma_C), \quad V := H^1(\Gamma_C), \quad W := \{\chi \in H^2(\Gamma_C) : \partial_{\mathbf{n}_s}\chi = 0\}, \end{gathered}$$

where we recall that

$$H^{1/2}_{00,\Gamma_D}(\Gamma_C;\mathbb{R}^3) = \left\{\mathbf{w} \in H^{1/2}(\Gamma_C;\mathbb{R}^3) : \exists\,\tilde{\mathbf{w}} \in H^{1/2}(\Gamma;\mathbb{R}^3) \text{ with } \tilde{\mathbf{w}} = \mathbf{w} \text{ in } \Gamma_C, \tilde{\mathbf{w}} = \mathbf{0} \text{ in } \Gamma_D\right\}. \tag{2.2}$$

The space $\mathbf{V}$ is endowed with the natural norm induced by $H^1(\Omega;\mathbb{R}^3)$. Throughout the paper, we will also use that

$$\mathbf{V} \subset L^4(\Gamma_C) \text{ continuously, and } \mathbf{V} \Subset L^{4-s}(\Gamma_C) \text{ compactly for all } s \in (0,3], \tag{2.3}$$

where the above embeddings have to be understood in the sense of traces.

Finally, we will use the symbols $c, c', C, C', \ldots$, with meaning possibly varying in the same line, to denote several positive constants only depending on known quantities. Analogously, with the symbols $I_1, I_2, \ldots$ we will denote several integral terms appearing in the estimates.

2.1. The bilinear forms in the momentum balance

We recall the definition of the standard bilinear forms of linear viscoelasticity, which are involved in the variational formulation of Equation (1.20a). Dealing with an anisotropic and inhomogeneous material, we assume that the fourth-order tensors $\mathbb{K} = (a_{ijkh})$ and $\mathbb{K}_v = (b_{ijkh})$, denoting the elasticity and the viscosity tensor, respectively, satisfy the classical symmetry and ellipticity conditions

$$\begin{aligned} &a_{ijkh} = a_{jikh} = a_{khij}, \qquad b_{ijkh} = b_{jikh} = b_{khij}, \quad i,j,k,h = 1,2,3,\\ &\exists\, \alpha_0 > 0: \qquad a_{ijkh}\xi_{ij}\xi_{kh} \ge \alpha_0 \xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij}: \xi_{ij} = \xi_{ji}, \quad i,j = 1,2,3,\\ &\exists\, \beta_0 > 0: \qquad b_{ijkh}\xi_{ij}\xi_{kh} \ge \beta_0 \xi_{ij}\xi_{ij} \quad \forall\, \xi_{ij}: \xi_{ij} = \xi_{ji}, \quad i,j = 1,2,3, \end{aligned} \tag{2.4a}$$

where the usual summation convention is used. Moreover, we require

$$a_{ijkh}, b_{ijkh} \in L^\infty(\Omega), \quad i,j,k,h = 1,2,3. \tag{2.4b}$$

By the previous assumptions on the elasticity and viscosity coefficients, the following bilinear forms $a, b : H^1(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^3) \to \mathbb{R}$, defined by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= \int_\Omega a_{ijkh}\epsilon_{kh}(\mathbf{u})\epsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3),\\ b(\mathbf{u},\mathbf{v}) &:= \int_\Omega b_{ijkh}\epsilon_{kh}(\mathbf{u})\epsilon_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3) \end{aligned}$$

turn out to be continuous and symmetric. In particular, we have

$$\exists\, M > 0: \ |a(\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{v})| \le M\|\mathbf{u}\|_{H^1(\Omega)}\|\mathbf{v}\|_{H^1(\Omega)} \quad \text{for all } \mathbf{u},\mathbf{v} \in H^1(\Omega;\mathbb{R}^3). \tag{2.5}$$

Moreover, since Γ_D has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\mathbf{V}$-elliptic, i.e. there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \ge C_a\|\mathbf{u}\|_{\mathbf{V}}^2, \qquad b(\mathbf{u},\mathbf{u}) \ge C_b\|\mathbf{u}\|_{\mathbf{V}}^2 \qquad \text{for all } \mathbf{u} \in \mathbf{V}. \tag{2.6}$$

2.2. Assumptions on the nonlinearities of the system

We will consider an extended version of system (1.20), where the subdifferentials $\partial I_{(-\infty,0]}$ in the boundary condition (1.20d) and in the flow rule (1.20e) for χ, and $\partial I_{[0,1]}$ in (1.20e), are replaced by more general subdifferential operators.

(1) We consider a function

$$\widehat{\alpha} : \mathbb{R} \to [0,+\infty] \quad \text{proper, convex, and lower semicontinuous, with } \widehat{\alpha}(0) = 0. \tag{2.7}$$

Note that as soon as $0 \in \mathrm{dom}(\widehat{\alpha})$, we can always reduce to the case $\widehat{\alpha}(0) = 0$ by a translation. Then, we introduce the proper, convex, and lower semicontinuous functional

$$\widehat{\boldsymbol{\alpha}} : \mathbf{Y} \to [0, +\infty] \quad \text{defined by} \quad \widehat{\boldsymbol{\alpha}}(\mathbf{u}) := \begin{cases} \int_{\Gamma_C} \widehat{\alpha}(\mathbf{u} \cdot \mathbf{n})\, dx & \text{if } \widehat{\alpha}(\mathbf{u} \cdot \mathbf{n}) \in L^1(\Gamma_C), \\ +\infty & \text{otherwise.} \end{cases}$$

We set $\boldsymbol{\alpha} := \partial \widehat{\boldsymbol{\alpha}} : \mathbf{Y} \rightrightarrows \mathbf{Y}'$. It follows from (2.7) that $\mathbf{0} \in \boldsymbol{\alpha}(\mathbf{0})$.

(2) We consider

$$\widehat{\rho} : \mathbb{R} \to [0, +\infty] \text{ proper, convex, and lower semicontinuous, with } \mathrm{dom}(\widehat{\rho}) \subset (-\infty, 0] \text{ and } \widehat{\rho}(0) = 0, \tag{2.8}$$

and set $\rho := \partial \widehat{\rho} : \mathbb{R} \rightrightarrows \mathbb{R}$.

(3) We let

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \text{ proper, convex, and lower semicontinuous, with } \mathrm{dom}(\widehat{\beta}) \subset [0, +\infty) \text{ and } \widehat{\beta}(0) = 0, \tag{2.9}$$

and set $\beta := \partial \widehat{\beta} : \mathbb{R} \rightrightarrows \mathbb{R}$.

The operator $\boldsymbol{\alpha} : \mathbf{Y} \rightrightarrows \mathbf{Y}'$ will replace $\partial I_{(-\infty,0]}$ in the boundary condition (1.20d). Thus, as soon as $\mathrm{dom}(\widehat{\alpha}) \subset (-\infty, 0]$, with $\boldsymbol{\alpha}$ we render the impenetrability (unilateral) constraint $\mathbf{u} \cdot \mathbf{n} \leq 0$ a.e. on $\Gamma_C \times (0, T)$. The operators ρ and β will generalize the subdifferentials $\partial I_{(-\infty,0]}$ and $\partial I_{[0,1]}$ in (1.20e). On the one hand, the requirement $\mathrm{dom}(\widehat{\beta}) \subset [0, +\infty)$ guarantees $\chi \geq 0$ a.e. in $\Gamma_C \times (0, T)$. On the other hand, starting from an initial datum χ_0 fulfilling $0 \leq \chi_0 \leq 1$ a.e. on Γ_C (cf. (2.12b) below) and taking into account that $\chi_t \leq 0$ a.e. in $\Gamma_C \times (0, T)$ since $\mathrm{dom}(\widehat{\rho}) \subset (-\infty, 0]$, we will ultimately deduce that $\chi \in [0, 1]$ a.e. on $\Gamma_C \times (0, T)$.

(4) As for the kernel k defining the operator $\mathcal{K}$ from (1.22), we will require that

$$k : \Gamma_C \times \Gamma_C \to [0, +\infty) \text{ is symmetric, with } k \in L^\infty(\Gamma_C \times \Gamma_C)\,. \tag{2.10}$$

(5) Finally, we will suppose that

$$\gamma \in C^2(\mathbb{R}) \quad \text{with } \gamma' \text{ Lipschitz on } \mathbb{R}. \tag{2.11}$$

2.3. Assumptions on the problem data

We suppose that

$$\mathbf{u}_0 \in \mathbf{V} \text{ with } \mathbf{u}_0 \in \mathrm{dom}(\widehat{\boldsymbol{\alpha}}), \tag{2.12a}$$

$$\chi_0 \in W, \quad \chi_0 \in [0, 1] \text{ on } \Gamma_C, \quad \beta^0(\chi_0) \in H, \tag{2.12b}$$

where $\beta^0(\chi_0)$ denotes the minimal section of $\beta(\chi_0)$. Then, from

$$0 \leq \widehat{\beta}(\chi_0) \leq \beta^0(\chi_0)\chi_0 \quad \text{a.e. on } \Gamma_C$$

(which follows from the positivity of $\widehat{\beta}$ and from the fact that $\widehat{\beta}(0) = 0$), we immediately deduce that

$$\widehat{\beta}(\chi_0) \in L^1(\Gamma_C).$$

As far as the body force $\mathbf{f}$ and the surface traction $\mathbf{h}$ are concerned, we prescribe that

$$\mathbf{f} \in L^2(0, T; \mathbf{H}), \tag{2.13a}$$

$$\mathbf{h} \in L^2(0, T; (H^{1/2}_{00,\Gamma_D}(\Gamma_N; \mathbb{R}^3))'), \tag{2.13b}$$

and we introduce $\mathbf{F} : (0,T) \to \mathbf{V}'$ by

$$\langle \mathbf{F}(t), \mathbf{v}\rangle_{\mathbf{V}} := \int_\Omega \mathbf{f}(t)\cdot \mathbf{v}\,\mathrm{d}x + \langle \mathbf{h}(t), \mathbf{v}\rangle_{H^{1/2}_{00,\Gamma_\mathrm{D}}(\Gamma_\mathrm{N};\mathbb{R}^3)} \quad \text{for all } \mathbf{v}\in\mathbf{V} \quad \text{for a.a. } t\in(0,T). \tag{2.14}$$

Of course, thanks to (2.13), $\mathbf{F}\in L^2(0,T;\mathbf{V}')$.

We are now in a position to give the variational formulation of the initial-boundary value problem for the (generalized) version of system (1.20) tackled in this paper.

Problem 2.2: Starting from initial data $(\mathbf{u}_0,\chi_0)$ fulfilling (2.12), find a quintuple $(\mathbf{u},\boldsymbol{\zeta},\chi,\omega,\xi)$ with

$$\mathbf{u}\in H^1(0,T;\mathbf{V}), \tag{2.15a}$$
$$\boldsymbol{\zeta}\in L^2(0,T;\mathbf{Y}'), \tag{2.15b}$$
$$\chi\in L^2(0,T;W)\cap L^\infty(0,T;V)\cap H^1(0,T;H), \tag{2.15c}$$
$$\omega,\ \xi\in L^2(0,T;H) \tag{2.15d}$$

such that $\mathbf{u}(0)=\mathbf{u}_0$, $\chi(0)=\chi_0$, and fulfilling a.e. in (0,T)

$$b(\mathbf{u}_t,\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_\mathrm{C}} \chi\mathbf{u}\mathbf{v}\,\mathrm{d}x + \langle\boldsymbol{\zeta},\mathbf{v}\rangle_{\mathbf{Y}} + \int_{\Gamma_\mathrm{C}} \chi\mathbf{u}\mathcal{K}[\chi]\mathbf{v}\,\mathrm{d}x = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{V}} \quad \text{for all } \mathbf{v}\in\mathbf{V}, \tag{2.16a}$$
$$\boldsymbol{\zeta}\in\boldsymbol{\alpha}(\mathbf{u}) \quad \text{in } \mathbf{Y}', \tag{2.16b}$$
$$\chi_t + \omega - \Delta\chi + \xi + \gamma'(\chi) = -\frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2}|\mathbf{u}|^2\mathcal{K}[\chi] - \frac{1}{2}\mathcal{K}[\chi|\mathbf{u}|^2] \quad \text{a.e. on } \Gamma_\mathrm{C}, \tag{2.16c}$$
$$\omega\in\rho(\chi_t) \quad \text{a.e. on } \Gamma_\mathrm{C}, \tag{2.16d}$$
$$\xi\in\beta(\chi) \quad \text{a.e. on } \Gamma_\mathrm{C}. \tag{2.16e}$$

Now we state the main result of this paper, which ensures the existence of solutions to Problem 2.2. Let us mention in advance that, for the selections $\omega\in\rho(\chi_t)$ and $\xi\in\beta(\chi)$ we will obtain stronger integrability properties than those required with (2.15d), cf. (2.17) below. They are obtained by means of a special a priori estimate, first introduced in [18], which is tailored to the doubly nonlinear structure of (2.16c) and has to be performed in order to separately estimate the selections in $\rho(\chi_t)$ and $\beta(\chi)$, cf. also Remark 3.2 ahead. It is in view of this estimate that the conditions $\chi_0\in W$ (in accordance with the property $\chi\in L^\infty(0,T;W)\cap W^{1,\infty}(0,T;H)\subset \mathrm{C}^0_{\mathrm{weak}}([0,T];W)$) and $\beta^0(\chi_0)\in H$ from (2.12b) are needed.

Theorem 2.1: *Assume (2.1), (2.4), (2.7)–(2.11), and (2.13). Then, for any pair $(\mathbf{u}_0,\chi_0)$ of initial data fulfilling (2.12) there exists a quintuple $(\mathbf{u},\boldsymbol{\zeta},\chi,\omega,\xi)$ solving Problem 2.2 and*

(1) *enjoying the enhanced integrability properties*

$$\chi\in L^\infty(0,T;W)\cap H^1(0,T;V)\cap W^{1,\infty}(0,T;H) \text{ and } \omega,\ \xi\in L^\infty(0,T;H), \tag{2.17}$$

and such that

$$0\le\chi(x,t)\le 1 \quad \text{for all } (x,t)\in\overline{\Gamma}_\mathrm{C}\times[0,T], \tag{2.18}$$

(2) *fulfilling the energy-dissipation inequality*

$$\int_s^t 2\mathcal{R}(\mathbf{u}_t(r),\chi_t(r))\,\mathrm{d}r + \int_s^t\int_{\Gamma_\mathrm{C}} \widehat{\rho}(\chi_t)\,\mathrm{d}x + \mathcal{E}(\mathbf{u}(t),\chi(t)) \le \mathcal{E}(\mathbf{u}(s),\chi(s)) + \int_s^t \langle\mathbf{F}(r),\mathbf{u}_t(r)\rangle_{\mathbf{V}}\,\mathrm{d}r \tag{2.19}$$

for all $0 \le s \le t \le T$, *with* $\mathcal{R}$ *and* $\mathcal{E}$ *given by* (2.21) *and* (2.22) *below.*
In fact, (2.19) *holds as a* balance *if* $\widehat{\rho}$ *is positively homogeneous of degree* 1 *(i.e.* $\widehat{\rho}(\lambda v) = \lambda\widehat{\rho}(v)$ *for all* $\lambda \ge 0$*), e.g. in the particular case* $\widehat{\rho} = I_{(-\infty,0]}$.

Remark 2.3: The energy-dissipation inequality (2.19) in fact holds along *any* solution to Problem 2.2. It reflects the fact that system (2.16) has a *(generalized) gradient system* structure. In fact, it can be rewritten as the (abstract) doubly nonlinear differential inclusion

$$\partial\mathcal{R}(\mathbf{u}_t(t), \chi_t(t)) + \partial^-\mathcal{E}(\mathbf{u}(t), \chi(t)) \ni 0 \qquad \text{in } \mathbf{V}' \times H \qquad \text{for a.a. } t \in (0,T), \tag{2.20}$$

involving the dissipation potential $\mathcal{R} : \mathbf{V} \times H \to [0, +\infty]$

$$\mathcal{R}(\mathbf{u}_t, \chi_t) := \mathcal{R}_{\mathsf{u}}(\mathbf{u}_t) + \mathcal{R}_\chi(\chi_t), \qquad \text{with } \begin{cases} \mathcal{R}_{\mathsf{u}}(\mathbf{u}_t) := \frac{1}{2}b(\mathbf{u}_t, \mathbf{u}_t), \\ \mathcal{R}_\chi(\chi_t) := \frac{1}{2}\|\chi_t\|_H^2, \end{cases} \tag{2.21}$$

and the energy functional

$$\begin{aligned} &\mathcal{E}(\mathbf{u}, \chi) := \mathcal{E}_1(\mathbf{u}) + \mathcal{E}_2(\mathbf{u}, \chi) \qquad \text{with} \\ &\qquad \mathcal{E}_1(\mathbf{u}) = \frac{1}{2}a(\mathbf{u}, \mathbf{u}) + \widehat{\boldsymbol{\alpha}}(\mathbf{u}), \\ &\qquad \mathcal{E}_2(\mathbf{u}, \chi) := \frac{1}{2}\int_{\Gamma_C} \chi|\mathbf{u}|^2\,dx + \frac{1}{2}\int_{\Gamma_C} \chi|\mathbf{u}|^2\mathcal{K}[\chi]\,dx + \int_{\Gamma_C}\left(\frac{1}{2}|\nabla\chi|^2 + \widehat{\beta}(\chi) + \gamma(\chi)\right)dx\,. \end{aligned} \tag{2.22}$$

In (2.20), $\partial\mathcal{R} : \mathbf{V} \times H \rightrightarrows \mathbf{V}' \times H$ is the subdifferential of $\mathcal{R}$ in the sense of convex analysis, while $\partial^-\mathcal{E} : \mathbf{V} \times H \rightrightarrows \mathbf{V}' \times H$ is the *Fréchet subdifferential* (cf., e.g. [19]) of $\mathcal{E}$. We will not specify its definition, here, but only mention that, in the case of the specific energy functional driving system (2.16), $\partial^-\mathcal{E}$ is given by the sum of the convex analysis subdifferentials of the convex contributions to $\mathcal{E}$, with the Gâteau-derivatives of the nonconvex, but smooth contributions. Thus, (2.20) yields (2.16).

However, for technical reasons, we will not directly exploit the structure (2.20) in the proof of our existence result for system (2.16). Nonetheless, (2.20) underlies (2.19).

Remark 2.4: No uniqueness result seems to be available for Problem 2.2, due to the doubly nonlinear character of the flow rule for χ. Nonetheless, arguing in the very same way as in the proof of [11, Proposition 2.3] (cf. also the proof of Proposition 4.1 ahead), it should be possible to prove a *local-in-time* uniqueness result, in the particular (physical) case of $\beta = \partial I_{[0,1]}$. Namely, that any two quintuples of solutions $(\mathbf{u}_1, \boldsymbol{\zeta}_1, \chi_1, \omega_1, \xi_1)$ and $(\mathbf{u}_2, \boldsymbol{\zeta}_2, \chi_2, \omega_2, \xi_2)$ such that there exists $T_0 \in (0, T]$ with

$$0 < \chi_i(x,t) < 1 \qquad \text{for all } (x,t) \in \overline{\Gamma}_C \times [0, T_0] \quad \text{for } i = 1, 2,$$

do coincide on $[0, T_0]$. In turn, the above separation property is guaranteed as soon as the initial datum χ_0 fulfills $\chi_0 \in [\delta, 1)$ on Γ_C, for some $\delta \in (0,1)$ (note that (2.12b) allows for χ_0 taking the values 0 and 1, instead).

The *proof* of Theorem 2.1 will be developed throughout Sections 3–5 by setting up an approximate system and proving the existence (and uniqueness) of local-in-time solutions for it (Section 3), passing to the limit with the approximation parameter and concluding the existence of local solutions for Problem 2.2 (Section 4), extending the local solution to a global one via a non-standard prolongation argument (Section 5).

3. The approximate system

After introducing the approximate problem in Sections 3.1 and 3.2, we prove its local-in-time well posedness by means of a fixed point argument combined with continuous dependence estimates.

3.1. Setup of the approximate problem

In system (3.2) below, we will replace the operators $\rho, \beta : \mathbb{R} \rightrightarrows \mathbb{R}$ featuring in the flow rule for χ by their Yosida regularizations ρ_ε and β_ε, cf. e.g. [20]. We will exploit that $\rho_\varepsilon, \beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions on $\mathbb{R}$, and denote by $\widehat{\rho}_\varepsilon$ and $\widehat{\beta}_\varepsilon$ their primitives fulfilling $\widehat{\rho}_\varepsilon(0) = 0$ and $\widehat{\beta}_\varepsilon(0) = 0$. The convex functions $\widehat{\rho}_\varepsilon$ and $\widehat{\beta}_\varepsilon$ are in fact the Yosida approximations of the functions $\widehat{\rho}$ and $\widehat{\beta}$, defined by

$$\widehat{\rho}_\varepsilon(y) := \min_{x\in\mathbb{R}} \left(\frac{1}{2\varepsilon}|y-x|^2 + \widehat{\rho}(x) \right) \qquad \widehat{\beta}_\varepsilon(y) := \min_{x\in\mathbb{R}} \left(\frac{1}{2\varepsilon}|y-x|^2 + \widehat{\beta}(x) \right). \tag{3.1}$$

We are now in a position to give the variational formulation of the approximate problem.

Problem 3.1: Starting from initial data $(\mathbf{u}_0, \chi_0)$ fulfilling (2.12), find a triple $(\mathbf{u}, \boldsymbol{\zeta}, \chi)$ as in (2.15a)–(2.15c) such that $\mathbf{u}(0) = \mathbf{u}_0$, $\chi(0) = \chi_0$, and fulfilling a.e. in (0,T)

$$b(\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_C} \chi \mathbf{u}\mathbf{v}\, dx + \langle \boldsymbol{\zeta}, \mathbf{v}\rangle_{\mathbf{Y}} + \int_{\Gamma_C} \chi \mathbf{u} \mathcal{K}[\chi]\mathbf{v}\, dx = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{V}} \quad \text{for all } \mathbf{v} \in \mathbf{V}, \tag{3.2a}$$

$$\boldsymbol{\zeta} \in \boldsymbol{\alpha}(\mathbf{u}) \quad \text{in } \mathbf{Y}', \tag{3.2b}$$

$$\chi_t + \rho_\varepsilon(\chi_t) - \Delta\chi + \beta_\varepsilon(\chi) + \gamma'(\chi) = -\frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2}|\mathbf{u}|^2\mathcal{K}[\chi] - \frac{1}{2}\mathcal{K}[\chi|\mathbf{u}|^2] \quad \text{a.e. on } \Gamma_C. \tag{3.2c}$$

Remark 3.2: On the one hand, with these regularizations, we will able to render all a priori estimates rigorously within the frame of the approximate system. In particular, we refer to a *regularity* estimate that consists in testing the flow rule (3.2c) by $\partial_t(-\Delta\chi + \beta_\varepsilon(\chi))$ (cf. the proof of Proposition 4.1). This estimate allows us to bound both $-\Delta\chi$ and (a selection) $\xi \in \beta(\chi)$ in $L^\infty(0, T; H)$, thus giving rise to (2.17) (since the estimates for χ_t and $\omega \in \rho(\chi_t)$ then follow by comparison in (2.16c)). On the other hand, since $\mathrm{dom}(\beta_\varepsilon) = \mathbb{R}$, in the approximate system (3.2) the constraint $\chi \geq 0$ will be no longer enforced. Because of this, the first a priori estimate performed on the approximate system (cf. again the proof of Proposition 4.1) will not have a *global-in-time* character, and the procedure for extending a local solution to a global one, developed in Section 5, will be more complex than the standard one.

3.2. Local-in-time existence for the approximate problem

This section is devoted to the proof of the following result.

Proposition 3.3: *Assume* (2.1), (2.4), (2.7)–(2.11), *and* (2.13). *Let* $(\mathbf{u}_0, \chi_0)$ *be a pair of initial data fulfilling* (2.12). *Then, there exists a final time* $\widehat{T} > 0$ *such that, for every* $\varepsilon > 0$, *Problem* 3.1 *admits a unique solution* $(u, \boldsymbol{\zeta}, \chi)$ *on* $(0, \widehat{T})$, *with the enhanced regularity* $\chi \in L^\infty(0, \widehat{T}; W) \cap H^1(0, \widehat{T}; V) \cap W^{1,\infty}(0, \widehat{T}; H)$.

We will prove Proposition 3.3 by constructing an operator, defined between suitable function spaces, whose fixed points yield solutions to system (3.2), and by showing that it does admit fixed points thanks to the Schauder theorem. This procedure will yield a local-in-time solution to Problem 3.1 defined on the interval $(0, \widehat{T})$, with $\widehat{T}$ in fact *independent* of ε, cf. Remark 3.8 ahead. In view of this, in Section 4 we will pass to the limit in system (3.2) as $\varepsilon \downarrow 0$ and obtain the existence of local-in-time solutions to the original Problem 2.2.

Our construction of the Schauder operator will be based on two results, Lemmas 3.5 and 3.6 ahead, tackling the separate solvability, on a given interval (0, T), of the momentum balance (3.2a), with χ

replaced by a *given* function $\overline{\chi} \in L^4(0,\mathsf{T};H)$, and of the flow rule (3.2c) for the adhesive parameter, with (the trace of) $\mathbf{u}$ replaced by a given $\overline{\mathbf{u}} \in H^1(0,\mathsf{T};L^4(\Gamma_C))$. Preliminarily, we fix the properties of the nonlocal operator $\mathcal{K}$ from (1.22) in the following result, whose proof is left to the reader.

Lemma 3.4: *Assume* (2.10). *Then, $\mathcal{K}$ is well defined, linear, and continuous from $L^1(\Gamma_C)$ to $L^\infty(\Gamma_C)$, with*

$$|\mathcal{K}[w](x)| \le \|k(x,\cdot)\|_{L^\infty(\Gamma_C)}\|w\|_{L^1(\Gamma_C)} \text{ for a.a. } x \in \Gamma_C, \text{ so that } \|\mathcal{K}[w]\|_{L^\infty(\Gamma_C)} \le \|k\|_{L^\infty(\Gamma_C\times\Gamma_C)}\|w\|_{L^1(\Gamma_C)}. \tag{3.3}$$

Furthermore, for every $1 \le p < \infty$ the operator $\mathcal{K}$ is continuous from $L^1(\Gamma_C)$, equipped with the weak topology, to $L^p(\Gamma_C)$ with the strong topology (i.e. if $w_n \rightharpoonup w$ in $L^1(\Gamma_C)$ then $\mathcal{K}[w_n] \to \mathcal{K}[w]$ in $L^p(\Gamma_C)$). Finally, there holds

$$\int_{\Gamma_C} \mathcal{K}[w_1](x)\, w_2(x)\,\mathrm{d}x = \int_{\Gamma_C} \mathcal{K}[w_2](x)\, w_1(x)\,\mathrm{d}x \qquad \text{for all } w_1,\, w_2 \in L^1(\Gamma_C)\,. \tag{3.4}$$

We start by tackling the momentum balance (3.2a). In what follows, we will denote by Q_i and $\widetilde{Q}_i$, $i = 1,\ldots$, computable, non-negative, and continuous functions, monotone increasing w.r.t. each of their variables, that will enter in the a priori estimates holding for the solutions to the momentum balance/adhesive flow rule.

Lemma 3.5: *Assume* (2.1), (2.4), (2.7), (2.10), *and* (2.13). *Let $\mathbf{u}_0$ fulfill* (2.12a). *Then, for every $\overline{\chi} \in L^4(0,\mathsf{T};H)$ there exists a unique pair $(\mathbf{u},\boldsymbol{\zeta}) \in H^1(0,\mathsf{T};\mathbf{V}) \times L^2(0,\mathsf{T};\mathbf{Y}')$ fulfilling $\mathbf{u}(0) = \mathbf{u}_0$ and*

$$b(\mathbf{u}_t,\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_C} \overline{\chi}\mathbf{u}\mathbf{v}\,\mathrm{d}x + \langle\boldsymbol{\zeta},\mathbf{v}\rangle_{\mathbf{Y}} + \int_{\Gamma_C} \overline{\chi}\mathbf{u}\mathcal{K}[\overline{\chi}]\mathbf{v}\,\mathrm{d}x = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{V}} \quad \text{for all } \mathbf{v} \in \mathbf{V} \tag{3.5}$$

for a.a. $t \in (0,\mathsf{T})$, with

$$\|\mathbf{u}\|_{H^1(0,\mathsf{T};\mathbf{V})} \le Q_1(\|\mathbf{u}_0\|_{\mathbf{V}}, \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0), \|\mathbf{F}\|_{L^2(0,\mathsf{T};\mathbf{V}')}, \|\overline{\chi}\|_{L^4(0,\mathsf{T};H)})\,. \tag{3.6}$$

Furthermore, there exists a positive function $\widetilde{Q}_1$ such that for every $\overline{\chi}_1, \overline{\chi}_2 \in L^4(0,\mathsf{T};H)$, with $\mathbf{u}_1$, $\mathbf{u}_2$ the associated solutions of (3.5) *starting from $\mathbf{u}_0$, there holds*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,\mathsf{T};\mathbf{V})} \le \widetilde{Q}_1\left(\max_{i=1,2}\|\mathbf{u}_i\|_{L^\infty(0,\mathsf{T};L^4(\Gamma_C))}, \max_{i=1,2}\|\chi_i\|_{L^4(0,\mathsf{T};H)}\right)\|\overline{\chi}_1 - \overline{\chi}_2\|_{L^4(0,\mathsf{T};H)}\,. \tag{3.7}$$

We denote by

$$\mathcal{T}_1 : L^4(0,\mathsf{T};H) \to H^1(0,\mathsf{T};\mathbf{V}) \quad \text{the solution operator associated with (3.5).} \tag{3.8}$$

Proof: A standard modification of the proof of [11, Proposition 4.2], which is in turn based on a time-discretization procedure, yields the existence statement. Estimate (3.6) follows from testing (3.5) by $\mathbf{u}_t$ and integrating on a time interval $(0,t)$, which gives

$$\begin{aligned} &\int_0^t b(\mathbf{u}_t,\mathbf{u}_t)\,\mathrm{d}r + \frac{1}{2}a(\mathbf{u}(t),\mathbf{u}(t)) + \widehat{\boldsymbol{\alpha}}(\mathbf{u}(t)) \\ &= \frac{1}{2}a(\mathbf{u}_0,\mathbf{u}_0) + \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0) + \int_0^t \langle\mathbf{F},\mathbf{u}_t\rangle_{\mathbf{V}}\,\mathrm{d}r - \int_0^t\int_{\Gamma_C} \overline{\chi}\mathbf{u}\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r - \int_0^t\int_{\Gamma_C} \overline{\chi}\mathbf{u}\mathcal{K}[\overline{\chi}]\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r \end{aligned} \tag{3.9}$$

thanks to the chain-rule formula

$$\int_0^t \langle\boldsymbol{\zeta},\mathbf{u}_t\rangle_{\mathbf{Y}}\,\mathrm{d}r = \widehat{\boldsymbol{\alpha}}(\mathbf{u}(t)) - \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0)\,. \tag{3.10}$$

In view of (2.6) and of the positivity of $\widehat{\boldsymbol{\alpha}}$, we have that

$$\text{left-hand side of (3.9)} \geq C_b \int_0^t \|\mathbf{u}_t\|_{\mathbf{V}}^2 \, \mathrm{d}r + \frac{C_a}{2} \|\mathbf{u}(t)\|_{\mathbf{V}}^2 .$$

Moreover, applying the Young inequality, we get

$$\left| \int_0^t \langle \mathbf{F}, \mathbf{u}_t \rangle_{\mathbf{V}} \, \mathrm{d}r \right| \leq \frac{C_b}{4} \int_0^t \|\mathbf{u}_t\|_{\mathbf{V}}^2 \, \mathrm{d}r + C \|\mathbf{F}\|_{L^2(0,\mathrm{T};\mathbf{V}')}^2 . \tag{3.11}$$

We then perform the following estimates for the last two terms on the right-hand side of (3.9)

$$\left| \int_0^t \int_{\Gamma_{\mathrm{C}}} \overline{\chi} \mathbf{u} \mathbf{u}_t \, \mathrm{d}x \, \mathrm{d}r \right| \leq \frac{C_b}{4} \int_0^t \|\mathbf{u}_t\|_{\mathbf{V}}^2 \, \mathrm{d}r + C \int_0^t \|\overline{\chi}\|_H^2 \|\mathbf{u}\|_{\mathbf{V}}^2 \, \mathrm{d}r,$$
$$\left| \int_0^t \int_{\Gamma_{\mathrm{C}}} \overline{\chi} \mathbf{u} \mathcal{K}[\overline{\chi}] \mathbf{u}_t \, \mathrm{d}x \, \mathrm{d}r \right| \leq \frac{C_b}{4} \int_0^t \|\mathbf{u}_t\|_{\mathbf{V}}^2 \, \mathrm{d}r + C \int_0^t \|\overline{\chi}\|_H^4 \|\mathbf{u}\|_{\mathbf{V}}^2 \, \mathrm{d}r,$$

where we have used (2.3), as well as estimate (3.3), yielding

$$\|\mathcal{K}[\chi]\|_{L^\infty(\Gamma_{\mathrm{C}})} \leq C \|\chi\|_{L^1(\Gamma_{\mathrm{C}})} \leq C \|\chi\|_H . \tag{3.12}$$

All in all, we obtain

$$\begin{aligned} &\frac{C_b}{4} \int_0^t \|\mathbf{u}_t\|_{\mathbf{V}}^2 \, \mathrm{d}r + \frac{C_a}{2} \|\mathbf{u}(t)\|_{\mathbf{V}}^2 \\ &\quad \leq C \left(\|\mathbf{u}_0\|_{\mathbf{V}}^2 + \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0) + \|\mathbf{F}\|_{L^2(0,\mathrm{T};\mathbf{V}')}^2 + \int_0^t \|\overline{\chi}\|_H^2 \|\mathbf{u}\|_{\mathbf{V}}^2 \, \mathrm{d}r + \int_0^t \|\overline{\chi}\|_H^4 \|\mathbf{u}\|_{\mathbf{V}}^2 \, \mathrm{d}r \right), \end{aligned}$$

whence (3.6) by the Gronwall Lemma.

In order to show the continuous dependence estimate (3.7), we subtract (3.5) for given $\overline{\chi}_2$ from (3.5) for $\overline{\chi}_1$ and test the resulting equation by $\mathbf{u}_1 - \mathbf{u}_2$. With calculations similar to those above, we obtain, by virtue of (3.3) and the Young inequality, that

$$\begin{aligned} &\frac{C_b}{2} \|(\mathbf{u}_1 - \mathbf{u}_2)(t)\|_{\mathbf{V}}^2 + \frac{C_a}{2} \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}^2 \, \mathrm{d}r \\ &\quad \leq \frac{C_a}{4} \int_0^t \|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{V}}^2 \, \mathrm{d}r + C \int_0^t \left(1 + \|\overline{\chi}_1\|_H + \|\overline{\chi}_1\|_H^2\right) \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^4(\Gamma_{\mathrm{C}})}^2 \, \mathrm{d}r \\ &\qquad + C \int_0^t \|\mathbf{u}_2\|_{L^4(\Gamma_{\mathrm{C}})}^2 (1 + \|\overline{\chi}_1\|_H^2 + \|\overline{\chi}_2\|_H^2) \|\overline{\chi}_1 - \overline{\chi}_2\|_H^2 \, \mathrm{d}r, \end{aligned} \tag{3.13}$$

whence (3.7). Finally, a comparison in (3.5) yields the uniqueness of $\boldsymbol{\zeta}$, too. □

We now address the flow rule (3.2c) for the adhesion parameter.

Lemma 3.6: *Assume (2.1), (2.8)–(2.11). Let χ_0 fulfill (2.12b). Then, for every $\overline{u} \in H^1(0,\mathsf{T}; L^4(\Gamma_{\mathrm{C}}; \mathbb{R}^3))$ there exists a unique $\chi \in L^\infty(0,\mathsf{T}; W) \cap H^1(0,\mathsf{T}; V) \cap W^{1,\infty}(0,\mathsf{T}; H)$ fulfilling $\chi(0) = \chi_0$ and*

$$\chi_t + \rho_\varepsilon(\chi_t) - \Delta\chi + \beta_\varepsilon(\chi) + \gamma'(\chi) = -\frac{1}{2}|\overline{\mathbf{u}}|^2 - \frac{1}{2}|\overline{\mathbf{u}}|^2 \mathcal{K}[\chi] - \frac{1}{2}\mathcal{K}[\chi|\overline{\mathbf{u}}|^2] \qquad \textit{a.e. on } \Gamma_{\mathrm{C}} \times (0,\mathsf{T}), \tag{3.14}$$

with

$$\|\chi\|_{L^2(0,\mathsf{T};W)\cap L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)} \le Q_2(\|\chi_0\|_V, \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}, \|\overline{\mathbf{u}}\|_{L^4(0,\mathsf{T};L^4(\Gamma_C))}) . \tag{3.15}$$

Furthermore, there exists a function $\widetilde{Q}_2$ *such that for every* $\overline{\mathbf{u}}_1,\ \overline{\mathbf{u}}_2 \in L^4(0,\mathsf{T};L^4(\Gamma_C;\mathbb{R}^3))$*, with* $\chi_1,\ \chi_2$ *the associated solutions to* (3.14) *emanating from the same initial datum* χ_0*, there holds*

$$\begin{aligned}&\|\chi_1-\chi_2\|_{L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)}\\ &\quad\le \widetilde{Q}_2\left(\varepsilon^{-1}, \max_{i=1,2}\|\overline{\mathbf{u}}_i\|_{L^4(0,\mathsf{T};L^4(\Gamma_C))}, \max_{i=1,2}\|\chi_i\|_{L^\infty(0,\mathsf{T};H)}\right)\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^2(0,\mathsf{T};L^4(\Gamma_C))}.\end{aligned} \tag{3.16}$$

We denote by

$$\begin{aligned}\mathcal{T}_2 : H^1(0,\mathsf{T};L^4(\Gamma_C;\mathbb{R}^3)) \to L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)\cap W^{1,\infty}(0,\mathsf{T};H)\\ \text{the solution operator associated with (3.14).}\end{aligned} \tag{3.17}$$

Proof: Observe that (3.14) can be recast as the abstract doubly nonlinear equation

$$\partial\mathcal{R}^\varepsilon_\chi(\chi_t(t)) + \partial^-\mathcal{E}_2(\overline{\mathbf{u}}(t),\chi(t)) \ni 0 \qquad \text{in } H \quad \text{for a.a. } t\in(0,\mathsf{T}), \tag{3.18}$$

with the dissipation potential $\mathcal{R}^\varepsilon_\chi(\dot\chi) := \frac{1}{2}\|\dot\chi\|_H^2 + \int_{\Gamma_C}\widehat{\rho}_\varepsilon(\dot\chi)\,\mathrm{d}x$ (cf. (2.21)), and the driving energy $\mathcal{E}_2$ from (2.22). In fact, here both the convex analysis subdifferential $\partial\mathcal{R}^\varepsilon_\chi$ and the Fréchet subdifferential $\partial^-\mathcal{E}_2$ reduce to singletons. Under the condition that $\overline{\mathbf{u}} \in H^1(0,\mathsf{T};L^4(\Gamma_C;\mathbb{R}^3))$, we may apply, for instance, [21, Theorem 2.2] to conclude the existence of a solution $\chi \in L^2(0,\mathsf{T};W)\cap L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)$ for (the Cauchy problem for) (3.18), via a time discretization procedure. The enhanced regularity $L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)\cap W^{1,\infty}(0,\mathsf{T};H)$ may be inferred by performing, on the time discrete level, the *regularity estimate* that consists in testing the flow rule by $\partial_t(-\Delta\chi+\beta_\varepsilon(\chi))$ (cf. the proof of Proposition 4.1).

Estimate (3.15) follows from testing (3.14) by χ_t and integrating in time. Taking into account that

$$\int_0^t\int_{\Gamma_C}\rho_\varepsilon(\chi_t)\chi_t\,\mathrm{d}x\,\mathrm{d}r \ge 0,$$

since $\rho_\varepsilon(0)=0$ and ρ_ε are increasing, and using that $\int_0^t\int_{\Gamma_C}\beta_\varepsilon(\chi)\chi_t = \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi(t))\,\mathrm{d}x - \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi_0)\,\mathrm{d}x$ by the chain rule, we get

$$\int_0^t\|\chi_t\|_H^2\,\mathrm{d}r + \frac{1}{2}\|\nabla\chi(t)\|_H^2 + \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi(t))\,\mathrm{d}x \le \frac{1}{2}\|\nabla\chi_0\|_H^2 + \int_{\Gamma_C}\widehat{\beta}(\chi_0)\,\mathrm{d}x + I_1+I_2+I_3+I_4,$$

where we have also used that $\int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi_0)\,\mathrm{d}x \le \int_{\Gamma_C}\widehat{\beta}(\chi_0)\,\mathrm{d}x$. With Young's inequality, we estimate

$$\begin{aligned}|I_1| &\le \int_0^t\int_{\Gamma_C}|\gamma'(\chi)||\chi_t|\,\mathrm{d}x\,\mathrm{d}r \overset{(1)}{\le} \int_0^t\int_{\Gamma_C}(|\gamma'(\chi_0)| + C|\chi-\chi_0|)|\chi_t|\,\mathrm{d}x\,\mathrm{d}r\\ &\qquad\le C\|\chi_0\|_H^2 + \frac{1}{8}\int_0^t\|\chi_t\|_H^2\,\mathrm{d}r + C\int_0^t\|\chi\|_H^2\,\mathrm{d}r + C,\\ |I_2| &\le \int_0^t\int_{\Gamma_C}|\overline{\mathbf{u}}|^2|\chi_t|\,\mathrm{d}x\,\mathrm{d}r \le \frac{1}{8}\int_0^t\|\chi_t\|_H^2\,\mathrm{d}r + 2\int_0^t\|\overline{\mathbf{u}}\|_{L^4(\Gamma_C)}^4\,\mathrm{d}r,\\ |I_3| &\le \int_0^t\int_{\Gamma_C}|\overline{\mathbf{u}}|^2|\mathcal{K}[\chi]||\chi_t|\,\mathrm{d}x\,\mathrm{d}r \le \frac{1}{8}\int_0^t\|\chi_t\|_H^2\,\mathrm{d}r + C\int_0^t\|\overline{\mathbf{u}}\|_{L^4(\Gamma_C)}^4\|\mathcal{K}[\chi]\|_{L^\infty(\Gamma_C)}^2\,\mathrm{d}r\end{aligned}$$

$$\overset{(2)}{\leq} \frac{1}{8}\int_0^t \|\chi_t\|_H^2\,\mathrm{d}r + C\int_0^t \|\overline{\mathbf{u}}\|_{L^4(\Gamma_C)}^4 \|\chi\|_H^2\,\mathrm{d}r,$$

$$|I_4| \leq \int_0^t\int_{\Gamma_C} |\mathcal{K}[\chi|\overline{\mathbf{u}}|^2]||\chi_t|\,\mathrm{d}x\,\mathrm{d}r \overset{(3)}{\leq} \frac{1}{8}\int_0^t \|\chi_t\|_H^2\,\mathrm{d}r + C\int_0^t \|\overline{\mathbf{u}}\|_{L^4(\Gamma_C)}^4 \|\chi\|_H^2\,\mathrm{d}r,$$

with (1) due to the Lipschitz continuity of γ', (2) to estimate (3.12), and (3) again following from (3.3), via $\|\mathcal{K}[\chi|\overline{\mathbf{u}}|^2]\|_{L^\infty(\Gamma_C)} \leq C\|\chi|\overline{\mathbf{u}}|^2\|_{L^1(\Gamma_C)} \leq C\|\overline{\mathbf{u}}\|_{L^4(\Gamma_C)}^2\|\chi\|_H$. Moreover, a comparison in (3.14) also provides a bound for $\|\Delta\chi\|_{L^2(0,\mathsf{T};H)}$, whence the estimate for χ in $L^2(0,\mathsf{T};W)$ by standard elliptic regularity results, relying on the assumption of $\mathrm{C}^{1,1}$-boundary for Γ_C.

Finally, in order to prove (3.16), let us preliminarily introduce, for fixed $\overline{\mathbf{u}} \in L^4(0,\mathsf{T};L^4(\Gamma_C;\mathbb{R}^3))$, the *Lipschitz continuous* mapping

$$\mathcal{F}(\overline{\mathbf{u}};\cdot): H \to H \text{ defined by } \mathcal{F}(\overline{\mathbf{u}};\chi) := -\beta_\varepsilon(\chi) - \gamma'(\chi) - \frac{1}{2}|\overline{\mathbf{u}}|^2 - \frac{1}{2}|\overline{\mathbf{u}}|^2\mathcal{K}[\chi] - \frac{1}{2}\mathcal{K}[\chi|\overline{\mathbf{u}}|^2].$$

Then, we subtract equation (3.14), written for a given $\overline{\mathbf{u}}_2$ in $L^4(0,\mathsf{T};L^4(\Gamma_C;\mathbb{R}^3))$, from (3.14) for $\overline{\mathbf{u}}_1$, test the resulting relation by $\partial_t(\chi_1-\chi_2)$. We use that

$$\int_{\Gamma_C} \left(\rho_\varepsilon(\partial_t\chi_1) - \rho_\varepsilon(\partial_t\chi_2)\right)(\partial_t\chi_1 - \partial_t\chi_2)\,\mathrm{d}x \geq 0$$

a.e. in $(0,\mathsf{T})$ by the monotonicity of ρ_ε, and, on the right-hand side, we estimate

$$\left|\int_{\Gamma_C} \left(\mathcal{F}(\overline{\mathbf{u}}_1;\chi_1) - \mathcal{F}(\overline{\mathbf{u}}_2;\chi_2)\right)\partial_t(\chi_1-\chi_2)\,\mathrm{d}x\right| \leq \frac{1}{2}\|\partial_t(\chi_1-\chi_2)\|_H^2 + \frac{1}{2}\|\mathcal{F}(\overline{\mathbf{u}}_1;\chi_1) - \mathcal{F}(\overline{\mathbf{u}}_2;\chi_2)\|_H^2.$$

All in all, we obtain

$$\begin{aligned} &\frac{1}{2}\|\partial_t(\chi_1-\chi_2)\|_H^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\chi_1-\chi_2)\|_H^2 \overset{(1)}{\leq} \frac{1}{2}\|\mathcal{F}(\overline{\mathbf{u}}_1;\chi_1) - \mathcal{F}(\overline{\mathbf{u}}_2;\chi_2)\|_H^2 \\ &\leq \left(C + \frac{1}{\varepsilon}\right)\|\chi_1-\chi_2\|_H^2 + C(\|\overline{\mathbf{u}}_1\|_{L^4(\Gamma_C)}^2 + \|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2)\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2 \\ &\quad + C\|\chi_1-\chi_2\|_H^2\|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^4 + C\|\chi_1\|_H^2(\|\overline{\mathbf{u}}_1\|_{L^4(\Gamma_C)}^2 + \|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2)\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2. \end{aligned} \tag{3.19}$$

a.e. in $(0,\mathsf{T})$. For (1), we have used the Lipschitz continuity of γ' and of β_ε (with Lipschitz constant $\frac{1}{\varepsilon}$), and we have for example estimated

$$\begin{aligned} &\||\overline{\mathbf{u}}_1|^2\mathcal{K}[\chi_1] - |\overline{\mathbf{u}}_2|^2\mathcal{K}[\chi_2]\|_H^2 \\ &\leq 2\|\overline{\mathbf{u}}_1+\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2\|\mathcal{K}[\chi_1]\|_{L^\infty(\Gamma_C)}^2 + 2\|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^4\|\mathcal{K}[\chi_1]-\mathcal{K}[\chi_2]\|_{L^\infty(\Gamma_C)}^2 \\ &\overset{(2)}{\leq} C\|\overline{\mathbf{u}}_1+\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2\|\chi_1\|_H^2 + C\|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^4\|\chi_1-\chi_2\|_H^2 \end{aligned}$$

where (2) follows from (3.3). Therefore, from (3.19) we deduce, via the Gronwall Lemma, that

$$\begin{aligned} \|\partial_t(\chi_1-\chi_2)(t)\|_H^2 \leq C\exp\left(C\left(1+\frac{1}{\varepsilon}\right)t + C\int_0^t \|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^4\right) \\ \times (\|\chi_1\|_H^2+1)(\|\overline{\mathbf{u}}_1\|_{L^4(\Gamma_C)}^2 + \|\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2)\|\overline{\mathbf{u}}_1-\overline{\mathbf{u}}_2\|_{L^4(\Gamma_C)}^2 \end{aligned}$$

for almost all $t \in (0,\mathsf{T})$. Upon integrating in time we then conclude estimate (3.16) for $\|\chi_1-\chi_2\|_{H^1(0,\mathsf{T};H)}$.

We then also recover the bound for $\|\chi_1-\chi_2\|_{L^\infty(0,\mathsf{T};V)}$. □

3.3. Solution operator for Problem 3.1 and application of the Schauder theorem

For fixed $M > 0$, we consider the closed ball

$$\mathcal{S}_M(\mathsf{T}) := \{\chi \in L^4(0,\mathsf{T};H) \,:\, \|\chi\|_{L^4(0,\mathsf{T};H)} \le M\}\,. \tag{3.20}$$

In view of (3.8) and (3.17), the operator

$$\mathcal{T} := \mathcal{T}_2 \circ \mathcal{T}_1 : L^4(0,\mathsf{T};H) \to L^\infty(0,\mathsf{T};W) \cap H^1(0,\mathsf{T};V) \cap W^{1,\infty}(0,\mathsf{T};H) \qquad \text{is well defined.} \tag{3.21}$$

Our next result shows that, at least for small times, the map $\mathcal{T}$ complies with the conditions of the Schauder theorem.

Lemma 3.7: *Assume (2.1), (2.4), (2.7)–(2.11), and (2.13). Let the initial data $(\mathbf{u}_0, \chi_0)$ fulfill (2.12). Then, there exists $\widehat{T} > 0$ such that the operator $\mathcal{T}$*

1. *maps $\mathcal{S}_M(\widehat{T})$ into itself;*
2. *is continuous with respect to the strong topology of $L^4(0,\widehat{T};H)$;*
3. *maps $\mathcal{S}_M(\widehat{T})$ into a compact subset of $L^4(0,\widehat{T};H)$.*

Proof:
Ad (1): Combining estimates (3.6) and (3.15), we find for every $\mathsf{T} \in (0,T]$

$$\|\mathcal{T}(\chi)\|_{H^1(0,\mathsf{T};H)\cap L^\infty(0,\mathsf{T};V)} \le Q_3(\|\mathbf{u}_0\|_{\mathbf{V}}, \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0), \|\chi_0\|_V, \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}, \|\mathbf{F}\|_{L^2(0,T;\mathbf{V}')}, \|\chi\|_{L^4(0,\mathsf{T};H)})\,. \tag{3.22}$$

Therefore,

$$\begin{aligned}\|\mathcal{T}(\chi)\|_{L^4(0,\mathsf{T};H)} &\\ \le C\mathsf{T}^{1/4}\|\mathcal{T}(\chi)\|_{L^\infty(0,\mathsf{T};V)} &\le \overline{C}\mathsf{T}^{1/4}Q_3(\|\mathbf{u}_0\|_{\mathbf{V}}, \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0), \|\chi_0\|_V, \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}, \|\mathbf{F}\|_{L^2(0,T;\mathbf{V}')}, M),\end{aligned}$$

so that property (1) follows by choosing

$$0 < \widehat{T} \le \left(\frac{M}{\overline{C}Q_3(\|\mathbf{u}_0\|_{\mathbf{V}}, \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0), \|\chi_0\|_V, \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}, \|\mathbf{F}\|_{L^2(0,T;\mathbf{V}')}, M)}\right)^4. \tag{3.23}$$

Ad (2): Let $(\chi_n)_n, \chi \subset \mathcal{S}_M(\widehat{T})$ fulfill $\chi_n \to \chi$ in $L^4(0,\widehat{T};H)$. Combining (3.6) with (3.7) we find that $\mathcal{T}_1(\chi_n) \to \mathcal{T}_1(\chi)$ in $L^\infty(0,\widehat{T};\mathbf{V})$. We then use (3.15) and (3.16) to conclude that $\mathcal{T}(\chi_n) = \mathcal{T}_2(\mathcal{T}_1(\chi_n)) \to \mathcal{T}(\chi) = \mathcal{T}_2(\mathcal{T}_1(\chi))$ in $L^\infty(0,\widehat{T};V) \cap H^1(0,\widehat{T};H)$, and thus in $L^4(0,\widehat{T};H)$.
Ad (3): The compactness property easily follows from combining estimates (3.6) and (3.15). □

Eventually, we are in a position to conclude the **proof of Proposition 3.3**. The existence statement follows from Lemma 3.7 via the Schauder theorem. Uniqueness of solutions to Problem 3.1 (the same argument would also yield continuous dependence on the initial and problem data) ensues by adding up estimates (3.13) and (3.19) and applying the Gronwall Lemma.

Remark 3.8: Estimate (3.23) shows that the local-existence time $\widehat{T}$ does not depend on ε.
A closer examination of the proof of Lemma 3.7 (based on Lemmas 3.5 and 3.6) in fact reveals that $\widehat{T}$ does not depend on the specific initial conditions $\mathbf{u}_0$, χ_0 either, but only on quantities related to them. In other words, for any 'ball' of initial data

$$\mathcal{I}(r) := \left\{(\mathbf{u}_0, \chi_0) \in \mathbf{V} \times W \,:\, \|\mathbf{u}_0\|_{\mathbf{V}} + \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0) + \|\chi_0\|_V + \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)} \le r\right\},$$

there exists a final time $\widehat{T}_r > 0$, only depending on r and on known constants, such that, for any $(\mathbf{u}_0, \chi_0) \in \mathcal{I}(r)$ and for any $\varepsilon > 0$, the approximate Problem 3.1, supplemented with the initial data $(\mathbf{u}_0, \chi_0)$ admits a solution on the interval $(0, \widehat{T}_r)$.

4. Local existence for Problem 2.2

Throughout this section, we will highlight the dependence of the local solution for Problem 3.1 (found in Proposition 3.3) on the approximation parameter ε, by denoting it with $(\mathbf{u}_\varepsilon, \boldsymbol{\zeta}_\varepsilon, \chi_\varepsilon)$. In this section, we perform an asymptotic analysis as $\varepsilon \downarrow 0$ of the sequence $(\mathbf{u}_\varepsilon, \boldsymbol{\zeta}_\varepsilon, \chi_\varepsilon)_\varepsilon$ on the existence interval $(0, \widehat{T})$, which *does not* depend on ε, cf. Remark 3.8.

We thus obtain the following existence result of *local-in-time* solutions for Problem 2.2.

Proposition 4.1: *Assume* (2.1), (2.4), (2.7)–(2.11), *and* (2.13). *Let* $(\mathbf{u}_0, \chi_0)$ *fulfill* (2.12). *Then, for every vanishing sequence* $\varepsilon_k \downarrow 0$ *as* $k \to \infty$ *there exists a quintuple* $(\widehat{\mathbf{u}}, \widehat{\boldsymbol{\zeta}}, \widehat{\chi}, \widehat{\omega}, \widehat{\xi})$, *with*

$$\widehat{\mathbf{u}} \in H^1(0, \widehat{T}; \mathbf{V}), \ \ \widehat{\chi} \in L^\infty(0, \widehat{T}; W) \cap H^1(0, \widehat{T}; V) \cap W^{1,\infty}(0, \widehat{T}; H),$$
$$\widehat{\boldsymbol{\zeta}} \in L^2(0, \widehat{T}; \mathbf{Y}'), \ \ \widehat{\omega} \in L^\infty(0, \widehat{T}; H), \ \ \widehat{\xi} \in L^\infty(0, \widehat{T}; H),$$

such that the following convergences hold as $k \to \infty$

$$\mathbf{u}_{\varepsilon_k} \rightharpoonup \widehat{\mathbf{u}} \qquad \text{in } H^1(0, \widehat{T}; \mathbf{V}), \tag{4.1a}$$
$$\mathbf{u}_{\varepsilon_k} \to \widehat{\mathbf{u}} \qquad \text{in } \mathrm{C}^0([0, \widehat{T}]; H^{1-s}(\Omega; \mathbb{R}^3)) \quad \text{for all } s \in (0,1), \tag{4.1b}$$
$$\mathbf{u}_{\varepsilon_k} \to \widehat{\mathbf{u}} \qquad \text{in } \mathrm{C}^0([0, \widehat{T}]; L^p(\Gamma_\mathrm{C}; \mathbb{R}^3)) \quad \text{for all } 1 \le p < 4, \tag{4.1c}$$
$$\chi_{\varepsilon_k} \overset{*}{\rightharpoonup} \widehat{\chi} \qquad \text{in } L^\infty(0, \widehat{T}; W) \cap H^1(0, \widehat{T}; V) \cap W^{1,\infty}(0, \widehat{T}; H), \tag{4.1d}$$
$$\chi_{\varepsilon_k} \to \widehat{\chi} \qquad \text{in } \mathrm{C}^0([0, \widehat{T}]; H^{2-s}(\Gamma_\mathrm{C})) \quad \text{for all } s \in (0,2), \tag{4.1e}$$
$$\boldsymbol{\zeta}_{\varepsilon_k} \rightharpoonup \widehat{\boldsymbol{\zeta}} \qquad \text{in } L^2(0, \widehat{T}; \mathbf{Y}'), \tag{4.1f}$$
$$\beta_{\varepsilon_k}(\chi_{\varepsilon_k}) \overset{*}{\rightharpoonup} \widehat{\xi} \qquad \text{in } L^\infty(0, \widehat{T}; H), \tag{4.1g}$$
$$\rho_{\varepsilon_k}(\partial_t \chi_{\varepsilon_k}) \overset{*}{\rightharpoonup} \widehat{\omega} \qquad \text{in } L^\infty(0, \widehat{T}; H)\,. \tag{4.1h}$$

Besides, $(\widehat{\mathbf{u}}, \widehat{\boldsymbol{\zeta}}, \widehat{\chi}, \widehat{\omega}, \widehat{\xi})$ *is a solution of Problem* 2.2, *fulfilling the energy-dissipation inequality* (2.19) *(holding as a* balance *if* $\widehat{\rho}$ *is* 1*-positively homogeneous), on* $(0, \widehat{T})$.

***Proof*:**
Step 1: a priori estimates. We derive some *a priori* estimates for (suitable norms of) the family $(\mathbf{u}_\varepsilon, \boldsymbol{\zeta}_\varepsilon, \chi_\varepsilon)_\varepsilon$ on the interval $(0, \widehat{T})$. First of all, recall that the solution components χ_ε are found as fixed points of the Schauder operator $\mathcal{T}$ from (3.21) in the ball $\mathcal{S}_M(\widehat{T})$. Therefore, in view of estimate (3.6) there exists a positive constant C, independent of ε, such that

$$\|\mathbf{u}_\varepsilon\|_{H^1(0,\widehat{T};\mathbf{V})} \le C\,. \tag{4.2}$$

Then, (3.15) (cf. also (3.22)) yields

$$\|\chi_\varepsilon\|_{L^2(0,\widehat{T};W) \cap L^\infty(0,\widehat{T};V) \cap H^1(0,\widehat{T};H)} \le C\,. \tag{4.3}$$

Secondly, we perform a comparison argument in (3.2a). Observe that, by the second of (3.3) and (4.3), we have

$$\|\mathcal{K}[\chi_\varepsilon]\|_{L^\infty((0,\widehat{T})\times\Gamma_\mathrm{C})} \le C\|\chi_\varepsilon\|_{L^\infty(0,\widehat{T};H)} \le C'. \tag{4.4}$$

Therefore, on account of (4.2)–(4.4) we obtain

$$\|\boldsymbol{\zeta}_\varepsilon\|_{L^2(0,\widehat{T};\mathbf{Y}')} \le C\,. \tag{4.5}$$

Next, we establish a further *regularity* estimate on χ_ε. We multiply (3.2c) by $\partial_t(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))$ and we integrate over $\Gamma_C\times(0,t)$, $t\in(0,\widehat{T})$. Observe that all the calculations below can be made rigorous, for the very same system (3.2), by arguing with difference quotients. We have

$$
\begin{aligned}
&\frac{1}{2}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|_H^2+\|\nabla\partial_t\chi_\varepsilon\|_{L^2(0,t;H)}^2+\int_0^t\int_{\Gamma_C}\beta_\varepsilon'(\chi_\varepsilon)(\partial_t\chi_\varepsilon)^2\,dx\,dr\\
&\quad+\int_0^t\int_{\Gamma_C}\rho_\varepsilon'(\partial_t\chi_\varepsilon)|\nabla(\partial_t\chi_\varepsilon)|^2\,dx\,dr+\int_0^t\int_{\Gamma_C}\beta_\varepsilon'(\chi_\varepsilon)\rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon\,dx\,dr\\
&\le\frac{1}{2}\|-\Delta\chi_0+\beta_\varepsilon(\chi_0)\|_H^2+I_1+I_2+I_3,
\end{aligned}\tag{4.6}
$$

where $I_i, i=1,2,3$, will be defined and estimated in what follows. We note that the integral terms on the left-hand side of (4.6) are non-negative, thanks to the monotonicity of ρ_ε and of β_ε and to the fact that $\rho_\varepsilon(0)=0$. Moreover, integrating by parts and taking into account (2.11), (2.12b), (4.2), and (4.3) we have

$$
\begin{aligned}
I_1&=-\int_0^t\int_{\Gamma_C}\gamma'(\chi_\varepsilon)\,\partial_t(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\,dx\,dr\\
&=\int_0^t\int_{\Gamma_C}\gamma''(\chi_\varepsilon)\partial_t\chi_\varepsilon\,(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\,dx\,dr-\int_{\Gamma_C}\gamma'(\chi_\varepsilon(t))(-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\,dx\\
&\quad+\int_{\Gamma_C}\gamma'(\chi_0)(-\Delta\chi_0+\beta_\varepsilon(\chi_0))\,dx\le c\int_0^t\|\partial_t\chi_\varepsilon\|_H\|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_H\,dr+c\|\chi_\varepsilon(t)\|_H^2\\
&\quad+\frac{1}{8}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|_H^2+c\|\chi_0\|_H\,\|-\Delta\chi_0+\beta_\varepsilon(\chi_0)\|_H+C\\
&\le c\int_0^t\|\partial_t\chi_\varepsilon\|_H\|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_H\,dr+\frac{1}{8}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|_H^2+C\,.
\end{aligned}\tag{4.7}
$$

Arguing in a similar way and using (2.12a), (2.12b), (3.3), (4.2), and (4.4), we infer

$$
\begin{aligned}
I_2&=-\frac{1}{2}\int_0^t\int_{\Gamma_C}|\mathbf{u}_\varepsilon|^2(1+\mathcal{K}[\chi_\varepsilon])\,\partial_t(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\,dx\,dr\\
&=\int_0^t\int_{\Gamma_C}\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon(1+\mathcal{K}[\chi_\varepsilon])\,(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\,dx\,dr\\
&\quad+\frac{1}{2}\int_0^t\int_{\Gamma_C}|\mathbf{u}_\varepsilon|^2\mathcal{K}[\partial_t\chi_\varepsilon]\,(-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon))\,dx\,dr\\
&\quad-\frac{1}{2}\int_{\Gamma_C}|\mathbf{u}_\varepsilon(t)|^2(1+\mathcal{K}[\chi_\varepsilon](t))\,(-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t)))\,dx\\
&\quad+\frac{1}{2}\int_{\Gamma_C}|\mathbf{u}_0|^2(1+\mathcal{K}[\chi_0])\,(-\Delta\chi_0+\beta_\varepsilon(\chi_0))\,dx\\
&\le c\int_0^t\|\mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)}\,\|\partial_t\mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)}\|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_H\,dr\\
&\quad+c\int_0^t\|\partial_t\chi_\varepsilon\|_{L^1(\Gamma_C)}\int_{\Gamma_C}|\mathbf{u}_\varepsilon|^2\,|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)|\,dx\,dr\\
&\quad+\frac{1}{8}\|-\Delta\chi_\varepsilon(t)+\beta_\varepsilon(\chi_\varepsilon(t))\|_H^2+c\|\mathbf{u}_\varepsilon(t)\|_{L^4(\Gamma_C)}^4+c\|\mathbf{u}_0\|_{L^4(\Gamma_C)}^2\,\|-\Delta\chi_0+\beta_\varepsilon(\chi_0)\|_H\\
&\le c\int_0^t\|\mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)}\,\|\partial_t\mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)}\|-\Delta\chi_\varepsilon+\beta_\varepsilon(\chi_\varepsilon)\|_H\,dr
\end{aligned}
$$

$$
\begin{aligned}
&+ c \int_0^t \|\partial_t \chi_\varepsilon\|_{L^1(\Gamma_C)} \|\mathbf{u}_\varepsilon\|^2_{L^4(\Gamma_C)} \| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_H \, dr \\
&+ \frac{1}{8}\| - \Delta\chi_\varepsilon(t) + \beta_\varepsilon(\chi_\varepsilon(t))\|^2_H + c \\
\le\; & c\, \|\mathbf{u}_\varepsilon\|_{L^\infty(0,\widehat{T};\mathbf{W})} \int_0^t \|\partial_t \mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)} \| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_H \, dr \\
&+ c\, \|\mathbf{u}_\varepsilon\|^2_{L^\infty(0,\widehat{T};\mathbf{W})} \int_0^t \|\partial_t \chi_\varepsilon\|_{L^1(\Gamma_C)} \| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_H \, dr \\
&+ \frac{1}{8}\| - \Delta\chi_\varepsilon(t) + \beta_\varepsilon(\chi_\varepsilon(t))\|^2_H + c.
\end{aligned}
$$

Prior to estimating I_3, we observe that, again by the second of (3.3) combined with (4.2)–(4.3), we have

$$
\|\mathcal{K}[\chi_\varepsilon |\mathbf{u}_\varepsilon|^2]\|_{L^\infty((0,\widehat{T})\times\Gamma_C)} \le c\|\chi_\varepsilon\|_{L^\infty(0,\widehat{T};H)} \, \|\mathbf{u}_\varepsilon\|^2_{L^\infty(0,\widehat{T};\mathbf{W})} \le C\,. \tag{4.8}
$$

Then, exploiting (2.12a), (2.12b), (3.3), (4.2)–(4.3), and (4.8), we get

$$
\begin{aligned}
I_3 =\; & -\frac{1}{2}\int_0^t \int_{\Gamma_C} \mathcal{K}[\chi_\varepsilon \,|\mathbf{u}_\varepsilon|^2]\, \partial_t(- \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon))\, dx\, dr \\
=\; & \int_0^t \int_{\Gamma_C} \mathcal{K}[\chi_\varepsilon \,\mathbf{u}_\varepsilon \cdot \partial_t \mathbf{u}_\varepsilon])\, (- \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon))\, dx\, dr \\
&+ \frac{1}{2}\int_0^t \int_{\Gamma_C} \mathcal{K}[\partial_t\chi_\varepsilon \,|\mathbf{u}_\varepsilon|^2]\, (- \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon))\, dx\, dr \\
&- \frac{1}{2}\int_{\Gamma_C} \mathcal{K}[\chi_\varepsilon \,|\mathbf{u}_\varepsilon|^2](t)\, (- \Delta\chi_\varepsilon(t) + \beta_\varepsilon(\chi_\varepsilon(t)))\, dx + \frac{1}{2}\int_{\Gamma_C} \mathcal{K}[\chi_0 \,|\mathbf{u}_0|^2]\, (- \Delta\chi_0 + \beta_\varepsilon(\chi_0))\, dx \\
\le\; & c \int_0^t \|\chi_\varepsilon\|_H \|\mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)} \, \|\partial_t \mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)} \int_{\Gamma_C} | - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)|\, dx\, dr \\
&+ c \int_0^t \|\partial_t\chi_\varepsilon\|_H \|\mathbf{u}_\varepsilon\|^2_{L^4(\Gamma_C)} \int_{\Gamma_C} | - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)|\, dx\, dr \\
&+ \frac{1}{8}\| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon(t))\|^2_H + \|\chi_0\|_H \, \|\mathbf{u}_0\|^2_{\mathbf{W}} \, \| - \Delta\chi_0 + \beta_\varepsilon(\chi_0)\|_{L^1(\Gamma_C)} + C \\
\le\; & c\|\chi_\varepsilon\|_{L^\infty(0,\widehat{T};H)} \, \|\mathbf{u}_\varepsilon\|_{L^\infty(0,\widehat{T};\mathbf{W})} \int_0^t \|\partial_t \mathbf{u}_\varepsilon\|_{L^4(\Gamma_C)} \| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_H \, dr \\
&+ c\|\mathbf{u}_\varepsilon\|^2_{L^\infty(0,\widehat{T};\mathbf{W})} \int_0^t \|\partial_t\chi_\varepsilon\|_H \| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_H \, dr + \frac{1}{8}\| - \Delta\chi_\varepsilon(t) + \beta_\varepsilon(\chi_\varepsilon(t))\|^2_H + C\,.
\end{aligned}
\tag{4.9}
$$

We plug the above estimates for $I_i, i = 1, 2, 3$ into (4.6), take into account the previously obtained (4.2), (4.3), and apply the Gronwall Lemma. In this way, we conclude that

$$
\| - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,\widehat{T};H)} \le C\,. \tag{4.10}
$$

On the other hand, by the monotonicity of β_ε, we have for a.e. $t \in (0, \widehat{T})$

$$
\|(- \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon))(t)\|^2_H \ge \| - \Delta\chi_\varepsilon(t)\|^2_H + \|\beta_\varepsilon(\chi_\varepsilon)(t)\|^2_H.
$$

Hence, by (4.3) and the aforementioned elliptic regularity results valid on Γ_C, we deduce that

$$\|\chi_\varepsilon\|_{H^1(0,\widehat{T};V)\cap L^\infty(0,\widehat{T};W)} \leq C \tag{4.11}$$

and, besides,

$$\|\beta_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,\widehat{T};H)} \leq C, \tag{4.12}$$

for some positive constant c independent of ε. Finally, a comparison in (3.2c), on account of (4.2), (4.11), and (4.12), entails

$$\|\rho_\varepsilon(\partial_t\chi_\varepsilon)\|_{L^\infty(0,\widehat{T};H)} \leq C\,. \tag{4.13}$$

Step 2: compactness argument. Thanks to the above estimates, by well-known weak and weak* compactness results, we deduce that, for every vanishing sequence $(\varepsilon_k)_k$ there exists a (not relabeled) subsequence of $\{(\mathbf{u}_{\varepsilon_k}, \boldsymbol{\zeta}_{\varepsilon_k}, \chi_{\varepsilon_k})\}_k$ such that convergences (4.1a), (4.1d), and (4.1f)–(4.1h) hold as $\varepsilon_k \downarrow 0$. Moreover, using Aubin–Lions compactness arguments and the generalized Ascoli theorem (cf., e.g. [22]), we also obtain (4.1b), (4.1c) and (4.1e), entailing that the pair $(\widehat{\mathbf{u}}, \widehat{\chi})$ (see Proposition 4.1) fulfills the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $\chi(0) = \chi_0$.

Step 3: limit passage in the approximate system (3.2). We exploit convergences (4.1).

To simplify notation, we omit the subindex k and denote the sequences of approximate solutions directly by the index ε. We first discuss the convergence of the non-local terms $\mathcal{K}[\chi_\varepsilon]$ and $\mathcal{K}[\chi_\varepsilon|\mathbf{u}_\varepsilon|^2]$. We mainly exploit the continuity properties of the operator $\mathcal{K}$ stated by Lemma 3.4. We first observe that (4.1e) implies that $\chi_\varepsilon \to \widehat{\chi}$ in $\mathrm{C}^0([0,\widehat{T}]; L^1(\Gamma_C))$, so that we can deduce

$$\mathcal{K}[\chi_\varepsilon] \to \mathcal{K}[\widehat{\chi}] \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; L^\infty(\Gamma_C)). \tag{4.14}$$

We analogously proceed to deal with the term $\mathcal{K}[\chi_\varepsilon|\mathbf{u}_\varepsilon|^2]$. Indeed, (4.1c) and (4.1e) lead to the strong convergence $\chi_\varepsilon|\mathbf{u}_\varepsilon|^2 \to \widehat{\chi}|\widehat{\mathbf{u}}|^2$ in $\mathrm{C}^0([0,\widehat{T}]; L^1(\Gamma_C))$, since by Sobolev embeddings χ_ε strongly converges in $\mathrm{C}^0([0,\widehat{T}]; L^p(\Gamma_C))$ for any $1 \leq p < \infty$, while $|\mathbf{u}_\varepsilon|^2 \to |\widehat{\mathbf{u}}|^2$ in $\mathrm{C}^0([0,\widehat{T}]; L^q(\Gamma_C))$ for all $1 \leq q < 2$. Thus, we can deduce

$$\mathcal{K}[\chi_\varepsilon|\mathbf{u}_\varepsilon|^2] \to \mathcal{K}[\widehat{\chi}|\widehat{\mathbf{u}}|^2] \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; L^\infty(\Gamma_C)). \tag{4.15}$$

Finally, (4.14) and (4.1c) with (4.4), yield

$$|\mathbf{u}_\varepsilon|^2\mathcal{K}[\chi_\varepsilon] \rightharpoonup |\widehat{\mathbf{u}}|^2\mathcal{K}[\widehat{\chi}] \text{ in } L^2(0,\widehat{T};H), \quad |\mathbf{u}_\varepsilon|^2\mathcal{K}[\chi_\varepsilon] \to |\widehat{\mathbf{u}}|^2\mathcal{K}[\widehat{\chi}] \text{ in } L^2(0,\widehat{T};L^p(\Gamma_C)) \text{ for all } 1 \leq p < 2. \tag{4.16}$$

Analogously, we have that (4.14) and (4.1c), (4.1e) imply (at least) that

$$\chi_\varepsilon\mathbf{u}_\varepsilon\mathcal{K}[\chi_\varepsilon] \to \widehat{\chi}\widehat{\mathbf{u}}\mathcal{K}[\widehat{\chi}] \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; H). \tag{4.17}$$

Now, it is a standard matter to pass to the limit (weakly) in the momentum balance (3.2a) and in the flow rule (3.2c) and conclude that $(\widehat{\mathbf{u}}, \widehat{\chi}, \widehat{\boldsymbol{\zeta}}, \widehat{\xi}, \widehat{w})$ fulfill (2.16a) and (2.16c). Actually, to complete our proof it remains to identify $(\widehat{\boldsymbol{\zeta}}, \widehat{\xi}, \widehat{w})$. We point out that (4.1e) and (4.1g) lead to

$$\limsup_{\varepsilon\searrow 0} \int_0^t \int_{\Gamma_C} \beta_\varepsilon(\chi_\varepsilon)\chi_\varepsilon \,\mathrm{d}x\,\mathrm{d}r \leq \int_0^t \int_{\Gamma_C} \widehat{\xi}\widehat{\chi} \,\mathrm{d}x\,\mathrm{d}r,$$

for all $t \in (0,\widehat{T}]$, whence the identification $\widehat{\xi} \in \beta(\widehat{\chi})$ a.e. in $\Gamma_C \times (0,\widehat{T})$. Then, we aim to prove that

$$\widehat{\boldsymbol{\zeta}} \in \boldsymbol{\alpha}(\widehat{\mathbf{u}}) \qquad \text{in } \mathbf{Y}' \text{ a.e. in } (0,\widehat{T}), \tag{4.18}$$

$$\widehat{w} \in \rho(\partial_t\widehat{\chi}) \qquad \text{a.e. in } \Gamma_C \times (0,\widehat{T}). \tag{4.19}$$

We first prove that

$$\limsup_{\varepsilon\searrow 0}\int_0^t \langle \boldsymbol{\zeta}_\varepsilon, \mathbf{u}_\varepsilon\rangle_{\mathbf{Y}}\,\mathrm{d}r \le \int_0^t \langle \widehat{\boldsymbol{\zeta}}, \widehat{\mathbf{u}}\rangle_{\mathbf{Y}}\,\mathrm{d}r, \tag{4.20}$$

whence (4.18). Let us test (3.2a) by $\mathbf{u}_\varepsilon$ and integrate over $(0,t)$. This gives

$$\begin{aligned}\int_0^t \langle \boldsymbol{\zeta}_\varepsilon, \mathbf{u}_\varepsilon\rangle_{\mathbf{Y}}\,\mathrm{d}r &= -\frac{1}{2}b(\mathbf{u}_\varepsilon(t),\mathbf{u}_\varepsilon(t)) + \frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0) \\ &- \int_0^t a(\mathbf{u}_\varepsilon,\mathbf{u}_\varepsilon)\,\mathrm{d}r - \int_0^t\int_{\Gamma_C}\chi_\varepsilon|\mathbf{u}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}r - \int_0^t\int_{\Gamma_C}\chi_\varepsilon\mathbf{u}_\varepsilon^2\mathcal{K}[\chi_\varepsilon]\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\langle F,\mathbf{u}_\varepsilon\rangle\,\mathrm{d}r.\end{aligned} \tag{4.21}$$

Taking the lim sup as $\varepsilon\searrow 0$, exploiting the lower semicontinuity of the bilinear forms a and b, as well as (4.1a)–(4.1e), (4.15), and (3.4), we can infer that

$$\begin{aligned}\limsup_{\varepsilon\searrow 0}\int_0^t \langle \boldsymbol{\zeta}_\varepsilon, \mathbf{u}_\varepsilon\rangle_{\mathbf{Y}}\,\mathrm{d}r &\le -\frac{1}{2}b(\widehat{\mathbf{u}}(t),\widehat{\mathbf{u}}(t)) + \frac{1}{2}b(\mathbf{u}_0,\mathbf{u}_0) \\ - \int_0^t a(\widehat{\mathbf{u}},\widehat{\mathbf{u}})\,\mathrm{d}r - \int_0^t\int_{\Gamma_C}\widehat{\chi}|\widehat{\mathbf{u}}|^2\,\mathrm{d}x\,\mathrm{d}r &- \int_0^t\int_{\Gamma_C}\widehat{\chi}\widehat{\mathbf{u}}^2\mathcal{K}[\widehat{\chi}]\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\langle F,\widehat{\mathbf{u}}\rangle\,\mathrm{d}r = \int_0^t\langle \widehat{\boldsymbol{\zeta}},\widehat{\mathbf{u}}\rangle_{\mathbf{Y}}\,\mathrm{d}r,\end{aligned} \tag{4.22}$$

as desired. For the above inequality, we have in particular used

$$-\int_0^t\int_{\Gamma_C}\chi_\varepsilon\mathbf{u}_\varepsilon^2\mathcal{K}[\chi_\varepsilon]\,\mathrm{d}x\,\mathrm{d}r \to -\int_0^t\int_{\Gamma_C}\widehat{\chi}\widehat{\mathbf{u}}^2\mathcal{K}[\widehat{\chi}]\,\mathrm{d}x\,\mathrm{d}r$$

due to (3.4), (4.1c), (4.1e), and (4.15). Analogously we proceed by testing (3.2c) by $\partial_t\chi_\varepsilon$ and integrate over $(0,t)$ to prove that

$$\limsup_{\varepsilon\searrow 0}\int_0^t\int_{\Gamma_C}\rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}r \le \int_0^t\int_{\Gamma_C}\widehat{w}\partial_t\widehat{\chi}\,\mathrm{d}x\,\mathrm{d}r \tag{4.23}$$

whence (4.19). We have

$$\begin{aligned}\int_0^t\int_{\Gamma_C}\rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}r &= -\|\partial_t\chi_\varepsilon\|^2_{L^2(0,t;H)} - \frac{1}{2}\|\nabla\chi_\varepsilon(t)\|_H^2 + \frac{1}{2}\|\nabla\chi_0\|_H^2 \\ &- \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi_\varepsilon(t))\,\mathrm{d}x + \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi_0)\,\mathrm{d}x - \int_{\Gamma_C}\gamma(\chi_\varepsilon(t))\,\mathrm{d}x + \int_{\Gamma_C}\gamma(\chi_0)\,\mathrm{d}x \\ &+ \int_0^t\int_{\Gamma_C}\left(-\frac{1}{2}|\mathbf{u}_\varepsilon|^2 - \frac{1}{2}|\mathbf{u}_\varepsilon|^2\mathcal{K}[\chi_\varepsilon] - \frac{1}{2}\mathcal{K}[\chi_\varepsilon|\mathbf{u}_\varepsilon|^2]\right)\partial_t\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}r\,.\end{aligned} \tag{4.24}$$

First, we recall that by Mosco convergence of $\widehat{\beta}_\varepsilon$ to $\widehat{\beta}$, we have

$$\liminf_{\varepsilon\searrow 0}\int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi_\varepsilon(t))\,\mathrm{d}x \ge \int_{\Gamma_C}\widehat{\beta}(\widehat{\chi}(t))\,\mathrm{d}x. \tag{4.25}$$

Thus, taking the limsup as $\varepsilon \searrow 0$ of both sides of (4.24) and exploiting (2.11), (4.1a)–(4.1e), (4.15), (4.16) and the lower semicontinuity properties of the Lebesgue norms, we get

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0} \int_0^t \int_{\Gamma_C} \rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon \,dx\,dr \le -\|\partial_t\widehat{\chi}\|^2_{L^2(0,t;H)} - \frac{1}{2}\|\nabla\widehat{\chi}(t)\|^2_H + \frac{1}{2}\|\nabla\chi_0\|^2_H \\
&- \int_{\Gamma_C} \widehat{\beta}(\widehat{\chi}(t))\,dx + \int_{\Gamma_C}\widehat{\beta}(\chi_0)\,dx - \int_{\Gamma_C}\gamma(\widehat{\chi}(t))\,dx + \int_{\Gamma_C}\gamma(\chi_0)\,dx \qquad (4.26)\\
&+ \int_0^t\int_{\Gamma_C}\left(-\frac{1}{2}|\widehat{\mathbf{u}}|^2 - \frac{1}{2}|\widehat{\mathbf{u}}|^2\mathcal{K}[\widehat{\chi}] - \frac{1}{2}\mathcal{K}[\widehat{\chi}|\widehat{\mathbf{u}}|^2]\right)\partial_t\widehat{\chi}\,dx\,dr = \int_0^t\int_{\Gamma_C}\widehat{w}\partial_t\widehat{\chi}\,dx\,dr,
\end{aligned}$$

which gives (4.23).

The energy-dissipation inequality (2.19) follows from testing the momentum balance (2.16a) by $\mathbf{u}_t$, the flow rule (2.16c) by χ_t, integrating along the time interval $(0,\widehat{T})$, and adding the above relations. The key ingredients to deduce (2.19) are the chain rules $\int_s^t a(\mathbf{u},\mathbf{u}_t)\,dr = \frac{1}{2}a(\mathbf{u}(t),\mathbf{u}(t)) - \frac{1}{2}a(\mathbf{u}(s),\mathbf{u}(s))$, (3.10), and

$$\begin{aligned}
&\int_s^t\int_{\Gamma_C}\chi\mathbf{u}\mathbf{u}_t\,dx\,dr + \frac{1}{2}\int_s^t\int_{\Gamma_C}|\mathbf{u}|^2\chi_t\,dx\,dr = \frac{1}{2}\int_{\Gamma_C}\chi(t)|\mathbf{u}(t)|^2\,dx - \frac{1}{2}\int_{\Gamma_C}\chi(s)|\mathbf{u}(s)|^2\,dx,\\
&\int_s^t\int_{\Gamma_C}\chi\mathbf{u}\mathcal{K}[\chi]\mathbf{u}_t\,dx\,dr + \frac{1}{2}\int_s^t\int_{\Gamma_C}|\mathbf{u}|^2\mathcal{K}[\chi]\chi_t\,dx\,dr + \frac{1}{2}\int_s^t\int_{\Gamma_C}\mathcal{K}[\chi|\mathbf{u}|^2]\chi_t\,dx\,dr\\
&\stackrel{(1)}{=} \frac{1}{2}\int_{\Gamma_C}\chi(t)|\mathbf{u}(t)|^2\mathcal{K}[\chi(t)]\,dx - \frac{1}{2}\int_{\Gamma_C}\chi(s)|\mathbf{u}(s)|^2\mathcal{K}[\chi(s)]\,dx,\\
&\int_s^t\int_{\Gamma_C}\xi\chi_t\,dx\,dr = \int_{\Gamma_C}\widehat{\beta}(\chi(t))\,dx - \int_{\Gamma_C}\widehat{\beta}(\chi(s))\,dx,
\end{aligned}$$

where for (1) we have also used the symmetry property (3.4). In particular, observe the first and second chain-rule identities allow us to combine the contributions from the momentum balance with those from the adhesive flow rule. We also use that

$$\int_s^t\int_{\Gamma_C}\omega\chi_t\,dx\,dr \ge \int_s^t\int_{\Gamma_C}\widehat{\rho}(\chi_t)\,dx\,dr \qquad (4.27)$$

since $\widehat{\rho}(0) = 0$. Then, (2.19) follows with straightforward calculations.

Observe that, if $\widehat{\rho}$ is 1-positively homogeneous, (4.27) in fact holds with an *equality* sign. Therefore, (2.19) is valid as a *balance*. □

5. From a local to a global solution: conclusion of the proof of Theorem 2.1

The standard procedure for extending the local solution of Problem 2.2 found with Proposition 4.1 to a global one would involve deriving suitable a priori estimates on (local) solutions of Problem 2.2 on a generic interval $(0,\mathsf{T})$, $\mathsf{T}\in(0,T]$, and showing that such estimates are in fact *independent* of the final time T. This would allow us to consider some maximal extension of the local solution from Proposition 4.1 and conclude, by a classical contradiction argument, that it must be defined on the whole interval $(0,T)$.

In the present case, in order to carry out the maximal extension argument, it would be necessary to rely on a global-in-time estimate (1) in $\mathrm{C}^0([0,\mathsf{T}];\mathbf{V})$ for $\mathbf{u}$; (2) in $L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)\subset \mathrm{C}^0_{\mathrm{weak}}([0,\mathsf{T}];W)$ for χ, in accordance with the requirements $\mathbf{u}_0\in\mathbf{V}$ and $\chi_0\in W$ on the initial data. However, while for $\mathbf{u}$ a global estimate in $H^1(0,\mathsf{T};\mathbf{V})$ directly follows from the energy-dissipation inequality (2.19) (cf. Lemma 5.2 ahead), from (2.19) we can derive *global* estimates for χ only in the spaces $L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)$ (and, as a by product, cf. Lemma 3.6, in $L^2(0,\mathsf{T};W)$, as well). In turn, the enhanced estimate $L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)$ seems to be obtainable only upon performing the (formally written, here) test by $\partial_t(-\Delta\chi+\xi)$, with $\xi\in\beta(\chi)$. As shown by the proof of Proposition 4.1, such a test can be made rigorous only in the frame of the approximate system (3.2). Unfortunately, for (3.2) the basic 'energy' estimates for $\mathbf{u}$ in $H^1(0,\mathsf{T};\mathbf{V})$ and for χ in $L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)$ do not possess a global-in-time character, cf. Remark 3.2 and again the proof of Proposition 4.1, essentially because the lack of the positivity constraint on χ does not allow one to control from below the energy $\mathcal{E}(\mathbf{u}(t),\chi(t))$ on the left-hand side of (2.19).

These technical difficulties related to the extension procedure were already manifest in the framework of the adhesive contact system first investigated in [11], where nonlocal effects were disregarded. In Section 5 therein, to overcome them we have developed a careful prolongation argument, where the key idea is to extend the ($\mathbf{u}$- and χ-components of a) local solution $(\widehat{\mathbf{u}},\widehat{\boldsymbol{\zeta}},\widehat{\chi},\widehat{\omega},\widehat{\xi})$, along with its 'approximability properties'. The latter plays a key role in recovering the enhanced regularity estimate for the χ-component in $L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)$.

In what follows, we will adapt the argument from [11] to our own situation. To avoid overburdening the paper, we will explain the steps of the procedure but omit several details and often refer to the calculations in [11, Section 5], which could indeed be repeated with minor changes in the present framework.

5.1. Scheme of the proof of the extension procedure

For the intents and purposes of our extension argument, we slightly modify the terminology introduced in Section 2 and call *solution of Problem 2.2* on an interval $(0,\mathsf{T})$

$$\begin{aligned}\text{any pair } (\mathbf{u},\chi) \text{ with } \mathbf{u}\in H^1(0,\mathsf{T};\mathbf{V}) \text{ and } \chi\in L^\infty(0,\mathsf{T};W)\cap H^1(0,\mathsf{T};V)\cap W^{1,\infty}(0,\mathsf{T};H) \text{ such that}\\ \text{there exist } (\boldsymbol{\zeta},\omega,\xi)\in L^2(0,\mathsf{T};\mathbf{Y}')\times L^\infty(0,\mathsf{T};H)\times L^\infty(0,\mathsf{T};H)\\ \text{such that } (\mathbf{u},\boldsymbol{\zeta},\chi,\omega,\xi) \text{ solve system (2.16) on } (0,\mathsf{T});\end{aligned} \tag{5.1}$$

we will adopt the very same convention for the solutions of the approximate Problem 3.1.

Recall that, in Proposition 4.1 we have shown that, along a sequence $\varepsilon_k\downarrow 0$, the (unique) solutions $(\mathbf{u}_{\varepsilon_k},\chi_{\varepsilon_k})$ of Problem 3.1 (with regularization parameter ε_k) converge to a solution $(\widehat{\mathbf{u}},\widehat{\chi})$ of Problem 2.2 on the time interval $(0,\widehat{T})$. In the following lines, for a fixed $\mathsf{T}\in(0,T]$ we will denote by $(\mathbf{u}_{\varepsilon_k}^{\mathsf{T}},\chi_{\varepsilon_k}^{\mathsf{T}})$ the unique solution of Problem 3.1_{ε_k} on $(0,\mathsf{T})$: clearly, for $\mathsf{T}\geq\widehat{T}$, $(\mathbf{u}_{\varepsilon_k}^{\mathsf{T}},\chi_{\varepsilon_k}^{\mathsf{T}})$ is the (unique) extension of $(\mathbf{u}_{\varepsilon_k},\chi_{\varepsilon_k})$.

We are now in a position to define the solution concept for Problem 2.2 that we are going to extend on the whole interval $[0,T]$.

Definition 5.1: Let $\mathsf{T}\in(0,T]$. We call a solution (in the sense of (5.1)) $(\mathbf{u}^{\mathsf{T}},\chi^{\mathsf{T}})$ of Problem 2.2 on $(0,\mathsf{T})$ an *approximable solution* if there exists a (not-relabeled) subsequence of $(\varepsilon_k)_k$ such that the related sequence $(\mathbf{u}_{\varepsilon_k}^{\mathsf{T}},\chi_{\varepsilon_k}^{\mathsf{T}})_k$ of solutions of Problem 3.1_{ε_k} on $(0,\mathsf{T})$ converge to $(\mathbf{u}^{\mathsf{T}},\chi^{\mathsf{T}})$ in the sense

$$\|\mathbf{u}_{\varepsilon_k}^{\mathsf{T}}-\mathbf{u}^{\mathsf{T}}\|_{C^0([0,\mathsf{T}];\mathbf{V})}+\|\chi_{\varepsilon_k}^{\mathsf{T}}-\chi^{\mathsf{T}}\|_{C^0([0,\mathsf{T}];V)}\to 0\quad\text{as } k\to\infty\,. \tag{5.2}$$

It is immediate to check that, for all $\mathsf{T}\in(\widehat{T},T]$ the functions $(\mathbf{u}^{\mathsf{T}},\chi^{\mathsf{T}})$ are a proper extension of $(\widehat{\mathbf{u}},\widehat{\chi})$.

Let us introduce the set

$$\mathscr{T}:=\{\mathsf{T}\in(0,T]\ :\ \text{there exists an approximable solution } (\mathbf{u}^{\mathsf{T}},\chi^{\mathsf{T}}) \text{ of Problem 2.2 on } (0,\mathsf{T})\}.$$

The calculations from the proof of Propostion 4.1 reveal that $(\mathbf{u}^{\mathsf{T}},\chi^{\mathsf{T}})$ fulfill the energy-dissipation inequality (2.19) (as a balance if $\widehat{\rho}$ is 1-positively homogeneous) along any sub-interval $[s,t]\subset[0,\mathsf{T}]$.

Clearly, $\widehat{T} \in \mathscr{T}$, which is thus non-empty. We aim to show that

$$T^* := \sup \mathscr{T} = T\,. \tag{5.3}$$

In this way, we will conclude that $(\widehat{\mathbf{u}}, \widehat{\chi})$ extends to a global solution on (0,T), fulfilling (2.19).
The proof of (5.3) is in turn split in the following steps:

Step 1 we show that the basic *energy estimates* for approximable solutions $(\mathbf{u}^{\mathsf{T}}, \chi^{\mathsf{T}})$ hold with constants independent of the final time T;
Step 2 we deduce that the local solution $(\widehat{\mathbf{u}}, \widehat{\chi})$ admits an extension $(\mathbf{u}^*, \chi^*)$ on $(0, T^*)$, with $\mathbf{u}^* \in H^1(0, T^*; \mathbf{V})$ and $\chi \in L^2(0, T^*; W) \cap L^\infty(0, T^*; V) \cap H^1(0, T^*; H)$;
Step 3 we show that $(\mathbf{u}^*, \chi^*)$ is a solution of Problem 2.2 (in the sense of (5.1)) on $(0, T^*)$;
Step 4 we conclude (5.3) via a contradiction argument.

We conclude this section by addressing the single steps with slightly more detail.
Step 1 We have the following result.
Lemma 5.2: *Assume* (2.1), (2.4), (2.7)–(2.11), *and* (2.13). *Then, there exists a positive function Q_4 such that, for any $\mathsf{T} > 0$ and any solution $(\mathbf{u}, \chi)$ of Problem* 2.2 *on* $(0, \mathsf{T})$ *there holds*

$$\|\mathbf{u}\|_{H^1(0,\mathsf{T};\mathbf{V})} + \|\chi\|_{L^2(0,\mathsf{T};W)\cap L^\infty(0,\mathsf{T};V)\cap H^1(0,\mathsf{T};H)} \le Q_4(\|\mathbf{u}_0\|_{\mathbf{V}}, \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0), \|\chi_0\|_V, \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}, \|\mathbf{F}\|_{L^2(0,\mathsf{T};\mathbf{V}')})\,.$$

Sketch of the proof: We consider the energy-dissipation inequality (2.19) on the interval $(0, \mathsf{T})$ and observe that the last integral term on the right-hand side can be absorbed into the dissipative term $\int_0^{\mathsf{T}} b(\mathbf{u}_t, \mathbf{u}_t)\,\mathrm{d}r$ on the left-hand side, cf. (3.11). Therefore, taking into account that

$$|\mathcal{E}(\mathbf{u}_0, \chi_0)| \le C(\|\mathbf{u}_0\|_{\mathbf{V}} + \widehat{\boldsymbol{\alpha}}(\mathbf{u}_0) + \|\chi_0\|_V + \|\widehat{\beta}(\chi_0)\|_{L^1(\Gamma_C)}),$$

from (2.19) we gather the bounds

$$\sup_{t\in[0,\mathsf{T}]} |\mathcal{E}(\mathbf{u}(t), \chi(t))| + \int_0^{\mathsf{T}} 2\mathcal{R}(\mathbf{u}_t, \chi_t)\,\mathrm{d}r \le C,$$

whence the estimates for $\mathbf{u}$ and χ in $H^1(0, \mathsf{T}; \mathbf{V})$ and $L^\infty(0, \mathsf{T}; V) \cap H^1(0, \mathsf{T}; H)$. The estimate for χ in $L^2(0, \mathsf{T}; W)$ follows from a comparison for $-\Delta\chi$ in the flow rule for χ, cf. the proof of Lemma 3.6.
Step 2 We consider a family $(\mathbf{u}^{\mathsf{T}}, \chi^{\mathsf{T}})_{\mathsf{T}\in\mathscr{T}}$ of approximable solutions, and define

$$\widetilde{\mathbf{u}}_{\mathsf{T}} : [0, T^*] \to \mathbf{V} \quad \text{by } \widetilde{\mathbf{u}}_{\mathsf{T}}(t) := \begin{cases} \mathbf{u}^{\mathsf{T}}(t) & \text{if } t \in [0, \mathsf{T}], \\ \mathbf{u}^{\mathsf{T}}(\mathsf{T}) & \text{if } t \in (\mathsf{T}, T^*], \end{cases}$$

$$\widetilde{\chi}_{\mathsf{T}} : [0, T^*] \to V \quad \text{by } \widetilde{\chi}_{\mathsf{T}}(t) := \begin{cases} \chi^{\mathsf{T}}(t) & \text{if } t \in [0, \mathsf{T}], \\ \chi^{\mathsf{T}}(\mathsf{T}) & \text{if } t \in (\mathsf{T}, T^*]. \end{cases}$$

We consider a sequence $(\mathsf{T}_m)_m \subset \mathscr{T}$ with $\mathsf{T}_m \uparrow T^*$ and extract a (not) relabeled subsequence $(\widetilde{\mathbf{u}}_{\mathsf{T}_m}, \widetilde{\chi}_{\mathsf{T}_m})_m$ suitably converging to a pair $(\mathbf{u}^*, \chi^*)$ defined on $[0, T^*]$, which extends $(\widehat{\mathbf{u}}, \widehat{\chi})$.
Step 3 We show that $\chi^* \in L^\infty(0, T^*; W) \cap H^1(0, T^*; V) \cap W^{1,\infty}(0, T^*; H)$ and that $(\mathbf{u}^*, \chi^*)$ is a solution of Problem 2.2, in the sense that

$$\exists\, (\boldsymbol{\zeta}^*, \omega^*, \xi^*) \in L^2(0, T^*; \mathbf{Y}') \times L^\infty(0, T^*; H) \times L^\infty(0, T^*; H) \text{ s.t. } (\mathbf{u}^*, \boldsymbol{\zeta}^*, \chi^*, \omega^*, \xi^*) \text{ fulfill (2.16) on } (0, T^*).$$

The proof of enhanced regularity for χ^* and of the existence of the triple $(\boldsymbol{\zeta}^*, \omega^*, \xi^*)$ relies on the very notion of *approximable* solution. For each element of the sequence $(\widetilde{\mathbf{u}}_{\mathsf{T}_m}, \widetilde{\chi}_{\mathsf{T}_m})_m$ from Step 2, we pick a sequence of solutions to the approximate Problem 3.1 on $(0, \mathsf{T}_m)$, converging to $(\widetilde{\mathbf{u}}_{\mathsf{T}_m}, \widetilde{\chi}_{\mathsf{T}_m})$ for fixed $m \in \mathbb{N}$. With a diagonalization procedure, we extract a subsequence converging to $(\mathbf{u}^*, \chi^*)$. We

perform the *regularity* estimates leading to the enhanced properties of χ^*, and to the existence of the triple $(\boldsymbol{\zeta}^*, \omega^*, \xi^*)$, on the level of the approximate system, by repeating the very same calculations in the proof of Proposition 4.1. We refer the reader to [11, Section 5] for all the details of the argument.
Step 4 Suppose that $T^* < T$. To obtain a contradiction, it is sufficient to extend the approximable solution $(\mathbf{u}^*, \chi^*)$, defined on the interval $[0, T^*]$, to an approximable solution on the interval $[0, T^* + \eta]$ for some $\eta > 0$. The argument for this is completely analogous to the one developed in [11, Section 5] for the adhesive contact system without nonlocal effects. We once again refer to [11] for all details.

In this way, we deduce (5.3) and thus conclude the proof of Theorem 2.1.

Disclosure statement

No potential conflict of interest was reported by the authors.

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